Chapter 8

Problems

1. $P\{0 \le X \le 40\} = 1 - P\{\,|X - 20| > 20\} \ge 1 - 20/400 = 19/20$

2. (a) $P\{X \ge 85\} \le E[X]/85 = 15/17$

   (b) $P\{65 \le X \le 85) = 1 - P\{\,|X - 75| > 10\} \ge 1 - 25/100$

   (c) $P\left\{\left|\sum_{i=1}^{n} X_i / n - 75\right| > 5\right\} \le \dfrac{25}{25n}$ so need $n = 10$

3. Let Z be a standard normal random variable. Then,

$$P\left\{\left|\sum_{i=1}^{n} X_i / n - 75\right| > 5\right\} \approx P\{\,|Z| > \sqrt{n}\,\} \le .1 \text{ when } n = 3$$

4. (a) $P\left\{\sum_{i=1}^{20} X_i > 15\right\} \le 20/15$

   (b)
$$\begin{aligned} P\left\{\sum_{i=1}^{20} X_i > 15\right\} &= P\left\{\sum_{i=1}^{20} X_i > 15.5\right\} \\ &\approx P\left\{Z > \frac{15.5 - 20}{\sqrt{20}}\right\} \\ &= P\{Z > -1.006\} \\ &\approx .8428 \end{aligned}$$

7. $$P\left\{\sum_{i=1}^{100} X_i > 525\right\} \approx P\left\{N(0,1) > \frac{525 - 500}{\sqrt{(100 \times 25)}}\right\} = P\{N(0,1) > .5\} = .3085$$

   where the above uses that an exponential with mean 5 has variance 25.

8. If we let X_i denote the life of bulb i and let R_i be the time to replace bulb i then the desired probability is $P\left\{\sum_{i=1}^{100} X_i + \sum_{i=1}^{99} R_i \le 550\right\}$. Since $\sum X_i + \sum R_i$ has mean $100 \times 5 + 99 \times .25 =$ 524.75 and variance $2500 + 99/48 = 2502$ it follows that the desired probability is approximately equal to $P\{N(0, 1) \le [550 - 524.75]/(2502)^{1/2}\} = P\{N(0, 1) \le .505\} = .693$
   It should be noted that the above used that

$$\text{Var}(R_i) = \text{Var}\left(\frac{1}{2}\text{Unif}[0,1]\right) = 1/48$$

9. Use the fact that a gamma $(n, 1)$ random variable is the sum of n independent exponentials with rate 1 and thus has mean and variance equal to n, to obtain:

$$\begin{aligned} P\left\{\left|\frac{X - n}{n}\right| > .01\right\} &= P\left\{|X - n| / \sqrt{n} > .01\sqrt{n}\right\} \\ &\approx P\left\{|N(0,1)| > .01\sqrt{n}\right\} \\ &= 2P\left\{N(0,1) > .01\sqrt{n}\right\} \end{aligned}$$

   Now $P\{N(0, 1) > 2.58\} = .005$ and so $n = (258)^2$.

13. (a) $P\{\bar{X} > 80\} = P\left\{\frac{\bar{X}-74}{14/5} > 15/7\right\} \approx PPZ > 2.14\} \approx .0162$

(b) $P\{\bar{Y} > 80\} = P\left\{\frac{\bar{Y}-74}{14/8} > 24/7\right\} \approx P\{Z > 3.43\} \approx .0003$

(c) Using that $SD(\bar{Y}-\bar{X}) = \sqrt{196/64+196/25} \approx 3.30$ we have

$$P\{\bar{Y}-\bar{X} > 2.2\} = P\{\bar{Y}-\bar{X}\}/3.30 > 2.2/3.30\}$$
$$\approx P\{Z > .67\} \approx .2514$$

(d) same as in (c)

15. $P\left\{\sum_{i=1}^{10,000} X_i > 2,700,000\right\} \approx P\{Z \geq (2,700,000 - 2,400,000)/(800 \cdot 100)\} = P\{Z \geq 3.75\} \approx 0$

16. (a) Number AJ's jobs, let X_i be the time it takes to do job i, and let $X_A = \sum_{i=1}^{20} X_i$ be the time that it takes AJ to finish all 20 jobs. Because

$$E[X_A] = 20(50) = 1000, \qquad \text{Var}(X_A) = 20(100) = 2000$$

the central limit theorem gives that

$$P\{X_A \leq 900\} = P\left\{\frac{X_A - 1000}{\sqrt{2000}} \leq \frac{900-1000}{\sqrt{2000}}\right\}$$
$$\approx P\{Z \leq -2.236\}$$
$$= 1 - \Phi(2.236) = .013$$

(b) Similarly, if we let X_M be the time that it takes MJ to finish all of her 20 jobs, then by the central limit theorem X_M is approximately normal with mean and variance

$$E[X_M] = 20(52) = 1040, \qquad \text{Var}(X_M) = 20(225) = 4500$$

Thus

$$P\{X_M \leq 900\} = P\left\{\frac{X_M - 1040}{\sqrt{4500}} \leq \frac{900-1040}{\sqrt{4500}}\right\}$$
$$\approx P\{Z \leq -2.087\}$$
$$= 1 - \Phi(2.087) = 0.18$$

(c) Because the sum of independent normal random variables is also normal, $D \equiv X_M - X_A$ is approximately normal with mean and variance

$$E[D] = 1040 - 1000 = 40, \qquad \text{Var}(D) = 4500 + 2000 = 6500$$

Hence,

$$P\{D > 0\} = P\left\{\frac{D-40}{\sqrt{6500}} \geq \frac{-40}{\sqrt{6500}}\right\}$$
$$\approx P\{Z \geq -.4961\}$$
$$= \Phi(.4961) = .691$$

Thus even though AJ is more likely than not to finish earlier than MJ, MJ has the better chance to finish within 900 minutes.

Theoretical Exercises

2. $P\{D > \alpha\} = P\{\,|X - \mu| > \alpha\mu\} \le \frac{\varsigma^2}{\alpha^2\mu^2} = \frac{1}{\alpha^2 r^2}$