Math 30530: Introduction to Probability, Fall 2012

Midterm Exam I

Review questions - solutions

1. 30 snowblowers, of which 7 have defects, are sold to a hardware store. The store manager inspects a total of 6 of the snowblowers randomly.

   (a) What is the probability that he finds no defective snowblowers?
   Solution:
   $$\frac{\binom{7}{0}\binom{23}{6}}{\binom{30}{6}}$$

   (b) What is the probability that he finds at least one defective snowblower?
   Solution:
   $$1-\frac{\binom{7}{0}\binom{23}{6}}{\binom{30}{6}}$$

   (c) What is the probability that he finds exactly two defective snowblowers?
   Solution:
   $$\frac{\binom{7}{2}\binom{23}{4}}{\binom{30}{6}}$$

2. Professor Bunsen always starts his Alchemy 231 lecture course with one of the three great alchemical experiments: turning lead into gold (20% of all times that he teaches the course), brewing the elixir of life (40% of the times) and creating the Philosopher's stone (40% of the time). When he tries to turn lead into gold, the result always ends with a explosion; when he brews the elixir of life, there is a 50% chance of an explosion, and when he creates the Philosopher's stone, 8 times out of 10 there is an explosion. Dean Crawford wants to see which experiment Professor Bunsen will do this year, but he arrives late. If he see the lecture-hall filled with post-explosion smoke, what should he conclude is the probability that he has just missed a demonstration of brewing the elixir of life?

   Solution: E, L and P are the three experiments (elixir, lead, philosopher's); S stands for smoke.

$$\Pr(E|S)=\frac{\Pr(S|E)\Pr(E)}{\Pr(S|E)\Pr(E)+\Pr(S|L)\Pr(L)+\Pr(S|P)\Pr(P)}=\frac{(.5)(.4)}{(.5)(.4)+(1)(.2)+(.8)(.4)}$$

3. A certain component in a (shoddy) computer typically fails 30% of the time, causing the computer to break. To counteract this appalling problem, a hacker decides to install n copies of the component in parallel, in such a way that the computer only breaks if all n components fail at the same time. Assuming that component failures are independent of each other,

   (a) find the probability that the computer does not break if $n = 3$ and
   Solution: $1 - (.3)^3$

   (b) find the smallest value of n that should be chosen to ensure that the probability that the computer does not break is at least 98%.
   Solution: Want n so that $1 - (.3)^n \geq .98$; $n = 4$ will do, $n = 3$ will not

4. A committee of 3 people is chosen from a group of 4 women and 3 women, with all such committees equally likely to be chosen. Let X be the number of women on the chosen committee.

   (a) Compute the mass function of X.
   Solution:

$$p_X(x) = \begin{cases} \frac{\binom{4}{3}\binom{3}{0}}{\binom{7}{3}} & \text{if } x = 0 \\ \frac{\binom{4}{2}\binom{3}{1}}{\binom{7}{3}} & \text{if } x = 1 \\ \frac{\binom{4}{1}\binom{3}{2}}{\binom{7}{3}} & \text{if } x = 2 \\ \frac{\binom{4}{0}\binom{3}{3}}{\binom{7}{3}} & \text{if } x = 3 \\ 0 & \text{otherwise} \end{cases}$$

   (b) What is the expectation and variance of X?
   Solution:

$$\begin{aligned} E(X) &= 0.p_X(0) + 1.p_X(1) + 2.p_X(2) + 3.p_X(3) \\ E(X^2) &= 0^2.p_X(0) + 1^2.p_X(1) + 2^2.p_X(2) + 3^2.p_X(3) \\ \text{Var}(X) &= E(X^2) - (E(X))^2 \end{aligned}$$

   (c) What is the probability that X is at most 2?
   Solution: $p_X(0) + p_X(1) + p_X(2)$

5. (a) What is the probability that at least two of the six members of a family were born on a Saturday or Sunday? (Assume that all days of the week are equally likely as birth days.)
   Solution: $1 - \binom{6}{0}(2/7)^0(5/7)^6 - \binom{6}{1}(2/7)^1(5/7)^5$

   (b) From the set of all families with four children, a child is selected at random and found to be a boy. Let X be the number of brothers the boy has. Compute the mass function, expectation and variance of X.
   Solution: X is binomial with $n = 3$, $p = 1/2$, so we know its mass; $E(X) = 3/2$, $\text{Var}(X) = 3/4$

6. If four different fair dice are tossed, what is the probability that they will show four different numbers? (Here *fair dice* means a dice that shows each of its six faces with equal probability.)

   Solution: $(6)(5)(4)(3)(1/6)^4$

7. Ten points are placed on the circumference of a circle. Two of them are selected at random. What is the probability that the two points are adjacent?

   Solution: $10/\binom{10}{2}$

8. In how many arrangements of the letters "WEWILLWHALEONAIRFORCE" are the three W's adjacent?

   Solution: $19!/(3!3!2!2!2!2!)$

9. Experience suggests that when sent a survey out, the response rate among under-25's 30% and the response rate among over-45's is 55%. A survey is sent out to equal numbers of under-25's, over-45's and 26-to-44's, and there is a 40% response rate. Assuming that the response rates for this particular survey are typical, what do you think is the usual response rate from 26-to-44's?

   Solution: Solve for x in $(.3)/3 + x/3 + (.55)/3 = .4$; solution is $100x\%$

10. A woman gives birth to a child in the hospital's maternity ward. After a while, the child is brought into the hospital's nursery. Before the child is brought in, there are 5 boys and 10 girls in the nursery. Sometime after the child is brought in, a doctor walks into the nursery, picks a child at random, and notices that it's a boy. What's the probability that the woman gave birth to a boy? (You should assume that when a child is born, the probability that it is a boy is .5 and the probability that it is a girl is .5.)

    Solution: Bb stands for the newborn baby being a boy; Gb stands for the newborn baby being a girl; Db stands for doctor picking a boy.

$$\Pr(Bb|Db) = \frac{\Pr(Db|Bb)\Pr(Bb)}{\Pr(Db|Bb)\Pr(Bb) + \Pr(Db|Gb)\Pr(Gb)} = \frac{(6/16)(1/2)}{(6/16)(1/2) + (5/16)(1/2)}$$

11. 2% of all people who are qualified to apply for a position as administrative assistant at a mathematics department are familiar with the typesetting language LaTeX. How many qualified applicants should a department interview if it wants to be 50% sure that at least one of the applicants is familiar with LaTeX?

    Solution: Want n so that $(.98)^n \leq .5$, so $n = 35$

12. For any two events A and B, say whether each of the statements below are always true or sometimes false. If true, give a proof; if sometimes false, give an example based on the experiment of rolling a fair dice and observing the number that is rolled.

    (a) $P(A \cup B) \geq P(AB)$.
    Solution: True ($A \cap B \subseteq A \cup B$)

(b) $P(AB) \geq P(A \cup B)$.

Solution: False in general; e.g., $A = \{1, 2\}$, $B = \{2, 3\}$

(c) $P(AB^c) + P(A^cB) \leq 1$.

Solution: True ($A \cap B^c$ and $A^c \cup B$ are disjoint)

13. A bag contains 30 balls numbered 1 through 30. Seven balls are selected at random, one at a time, with replacement. What is the probability that exactly four of the selected balls have prime numbers on them?

Solution: There are 10 prime numbers between 1 and 30, so the probability is $\binom{7}{4}(1/3)^4(2/3)^3$

14. A fair coin (one that shows heads and tails each with probability 1/2) is tossed repeatedly until the first time that the same face comes up twice in a row. Let X be the random variable that counts the number of tosses needed until this happens.

(a) What are the possible values that X can take?

Solution: $2, 3, 4, 5, \ldots, \infty$

(b) Compute the mass function of X.

Solution:

$$p_X(x) = \begin{cases} 2(1/2)^x & \text{if } x = 2, 3, 4, 5, \ldots \\ 0 & \text{otherwise} \end{cases}$$

(c) What is the probability that it will take more than 3 tosses to first see the same face coming up twice in a row?

Solution: $1 - p_X(2) - p_X(3) = 1 - 2(1/2)^2 - 2(1/2)^3$

(d) Compute the expectation of X.

Solution: $X - 1$ is geometric with parameter $1/2$, so $E(X - 1) = 2$, so $E(X) = 3$