Math 30530: Introduction to Probability, Fall 2012

Midterm Exam II

Review problems with solutions

1. The time it takes for a calculus student to answer all the questions on a certain exam is an exponentially distributed random variable with mean 1 hour and 15 minutes. If 10 students are taking the exam, what is the probability that at least one of them completes it in less than 1 hour?

   Solution: Let X be the time it takes a particular student to finish. Since X is exponential with mean 1.25 (time measured in hours), it follows that $\lambda = .8$ (remember that $E(X) = 1/\lambda$). So the probability that the particular student finishes in less than 1 hour is

   $$P(X < 1) = \int_0^1 .8e^{-.8x}\,dx = [-e^{-.8x}]_{x=0}^{x=1} = 1 - e^{-.8} = .55....$$

   Assuming that the students' finishing times are independent, the probability that none of them finishes in less than one hour is

   $$(1 - (1 - e^{-.8}))^{10} = e^{-8} = 3.35... \times 10^{-4}.$$

   So the probability that at least one of them finishes in less than one hour is

   $$1 - e^{-8} = .99966....$$

2. The joint density of a pair of random variables X, Y is

   $$f(x, y) = \begin{cases} Cxe^{-4y} & \text{if } 0 \le x \le 4 \text{ and } 0 \le y \le \infty \\ 0 & \text{otherwise.} \end{cases}$$

   (a) Find C.

   Solution: We compute

   $$\int_{x=0}^{4} \int_{y=0}^{\infty} Cxe^{-4y}\,dydx = C \int_{x=0}^{4} x\,dx \int_{y=0}^{\infty} e^{-4y}\,dy = C[x^2/2]_0^4[-e^{-4y}/4]_0^\infty = 2C.$$

   Since we must have the double integral be 1, we must have $C = 1/2$.

   (b) Are X and Y independent? Give a brief explanation.

   Solution: Because the density function is non-zero on a rectangle, and on that rectangle $f(x, y)$ factors into terms involving x only and terms involving y only, the two random variable X and Y are independent.

(c) Write down (but don't evaluate) an integral whose value is the probability that $X + Y \geq 4$.

Solution:

$$\int_0^4 \int_{4-x}^{\infty} \frac{xe^{-4y}}{2}\, dydx.$$

(d) Compute the mean and variance of $XY^2 + X^2Y$ (just write down the integrals that you would compute).

Solution:

$$E(XY^2 + X^2Y) = \int_{x=0}^4 \int_{y=0}^{\infty} (xy^2 + x^2y)\frac{xe^{-4y}}{2}\, dydx,$$

$$E((XY^2 + X^2Y)^2) = \int_{x=0}^4 \int_{y=0}^{\infty} (xy^2 + x^2y)^2\frac{xe^{-4y}}{2}\, dydx$$

(and then $\text{Var}(XY^2 + X^2Y) = E((XY^2 + X^2Y)^2) - (E(XY^2 + X^2Y))^2$).

3. Let X and Y be independent random variables, each uniformly distributed on $[0, 1]$. Find the probability that the quadratic equation

$$t^2 - Xt + Y = 0$$

has real roots.

Solution: By the quadratic formula, the condition that ensures real roots is $X^2 - 4Y \geq 0$, or $Y \leq X^2/4$. The probability that this relation holds is just the area under the curve $y = x^2/4$ between $x = 0$ and $x = 1$. This is

$$\int_0^1 \frac{x^2}{4}\, dx = 1/12.$$

4. I arrive at the bus stop by the bookstore at noon. I know that the number of hours that will pass before the bus to Midway arrives is a random variable X that is uniformly distributed on $[0, 1/2]$. The National Weather service tells me that it will begin raining sometime after noon, and that the number of hours that will pass before the rain starts is an exponentially distributed random variable Y with $\lambda = 2$.

(a) Write down the joint density function $f(x, y)$ of X and Y.

Solution: Unless both $0 \leq x \leq 1/2$ and $0 \leq y \leq \infty$, the joint density $f(x, y)$ is 0. For $0 \leq x \leq 1/2$ and $0 \leq y < \infty$, we have

$$f(x, y) = 2 \times 2e^{-2y} = 4e^{-2y}.$$

(b) Compute the probability that it will begin to rain before the bus arrives.

Solution: We want $P(Y < X)$. The region of the plane in which the joint density is non-zero, and $y < x$, is the triangle with vertices $(0, 0)$, $(1/2, 0)$ and $(1/2, 1/2)$. So

$$P(Y < X) = \int_{x=0}^{1/2} \int_{y=0}^{x} 4e^{-2y}\, dydx = \int_0^{1/2} (2 - 2e^{-2x})\, dx = e^{-1}.$$

5. Let the random variables X and Y have joint density

$$f(x,y) = \begin{cases} \frac{1}{2}e^{-x}, & \text{if } x > 0 \text{ and } 0 < y < 2 \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find the marginal densities f_X and f_Y.

Solution: In general, $f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$ and $f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$. Applying here, we get

$$f_X(x) = \begin{cases} e^{-x}, & \text{if } x > 0 \\ 0, & \text{otherwise} \end{cases}$$

and

$$f_Y(y) = \begin{cases} \frac{1}{2} & \text{if } 0 < y < 2 \\ 0, & \text{otherwise.} \end{cases}$$

Note that to fully specify the marginal densities, we need to give their values for **all** possible inputs (including those intervals where the densities are 0).

(b) Are X and Y independent? Explain.

Solution: Because $f_X(x)f_Y(y) = f(x,y$ for all x, y, they are independent.

(c) Find $P\{X < Y\}$.

Solution: After drawing a diagram, we see that

$$P(X < Y) = \int 0^2 \int 0^y e^{-x}/2\,dxdy = \int 0^2 \int_x^2 e^{-x}/2\,dydx = (1 + e^{-2})/2.$$

(The first of these is much easier to calculate than the second.)

6. A box contains a large number of nails whose lengths are normally distributed with mean 4 cm and standard deviation .1 cm. 10 nails are chosen at random from the batch. What is the probability that exactly 5 of them will be between 3.9 cm and 4.1 cm in length?

Solution: The probability that a single nail is between 3.9 and 4.1 is $\Pr(3.9 < X < 4.1)$ where $X \sim \mathcal{N}(4, (.1)^2)$, which is the same as $\Pr(-1 \le Z \le 1)$, approximately .68. So the probability that exactly 5 of the nails satisfy the length condition is

$$\binom{10}{5}(.68)^5(.32)^5.$$

7. Dr. G. teaches on MWF 1.55-2:45 pm. On Mondays he usually makes a mistake, on average, every 10 minutes. On Wednesdays he mostly gets his act together, but still adds things incorrectly once about every 25 minutes. By Friday he gets tired and, on average, misspells a word eight times during the lecture period. On each day, the number of mistakes he makes is modeled by a Poisson random variable. If a randomly chosen lecture (equally likely to be M, W or F) happened to be surprisingly error free, what is the probability that it was delivered on Friday?

Solution: Let M, W, F be the random variables that model the number of mistakes made on each of the days, so $M \sim \text{Poisson}(5)$, $W \sim \text{Poisson}(2)$, $F \sim \text{Poisson}(8)$. Let E be the event that no errors are made. We want $\Pr(F|E)$. By Bayes' formula,

$$\Pr(F|E) = \frac{\Pr(E|F)\Pr(F)}{\Pr(E|M)\Pr(M) + \Pr(E|W)\Pr(W) + \Pr(E|F)\Pr(F)} = \frac{e^{-8}}{e^{-5} + e^{-2} + e^{-8}}.$$

8. A random variable X has density function given by

$$f(x) = \begin{cases} \frac{3}{x^4} & \text{if } x \geq 1 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Compute $E(X)$.

Solution:

$$E(X) = \int_1^\infty x\frac{3}{x^4}\, dx = 3\int_1^\infty x^{-3}\, dx = 3\left[-\frac{1}{2x^2}\right]_1^\infty = \frac{3}{2}.$$

(b) Compute the distribution function $F_{X^2}(t)$ of X^2. (Give its value for all t.) **Solution**: For $t < 1$ we have $F_{X^2}(t) = 0$. For $t \geq 1$,

$$F_{X^2}(t) = \Pr(X^2 \leq t) = \Pr(X \leq \sqrt{t}) = \int_1^{\sqrt{t}} \frac{3}{x^4}\, dx = \left[-\frac{1}{x^3}\right]_1^{\sqrt{t}} = 1 - \frac{1}{t^{3/2}}.$$

(c) Compute the density function $f_{X^2}(x)$ of X^2.

Solution: Differentiating the result from the last part, we get that $f_{X^2}(x) = 0$ if $x < 1$ and if $x \geq 1$:

$$f_{X^2}(x) = -\frac{3}{2}t^{-5/2}.$$