1. **GW 25.1**: The area of the triangle is $9/2$, so on the triangle the joint density is $2/9$; everywhere else it is 0. The region $R$ of the triangle where $X + Y > 2$ is shown shaded in figure 1 of the figures page, and $\Pr(X + Y \geq 2)$ is the double integral of the density over that region. Since the density is constant, this is just $2/9$ times the area of the region, or $(2/9) \times (5/2) = 5/9$.

2. **GW 25.2**: As with the previous question, we calculate the area of the shaded region in figure 2 of the figures page, divided by the total area; it’s $12/18$ or $2/3$.

3. **GW 25.8**: a) \[
\int_1^2 \int_3^4 \frac{1}{24}(x+1)(y^2+1) \, dy \, dx = \frac{25}{228} \approx .1109649.
\] 
b) \[
f_X(x) = \begin{cases} 0 & \text{if } x < 0 \text{ or } x > 4 \\ \frac{1}{12}(x+1) & \text{if } 0 \leq x \leq 4 \\ \end{cases}
\] 
c) \[
f_Y(y) = \begin{cases} 0 & \text{if } y < 0 \text{ or } y > 4 \\ \frac{3}{76}(y^2+1) & \text{if } 0 \leq y \leq 4 \\ \end{cases}
\]

4. **GW 25.20**: We have to do a double integral over the shaded region shown in figure 3 of the figures page, of the joint density. This is best done as an iterated integral with $x$ on the outside: \[
\int_{x=0}^{1/2} \int_{y=-1}^{2x} \left( \frac{1}{4} \cos x \sin y + \frac{1}{4} \right) \, dx \, dy = \frac{1}{48} \left( 9 - 12 \sin(1/2) + 4 \sin(3/2) \right) \approx .150768.
\]

5. **GW 26.4**: The marginal density of $X$ is \[
f_X(x) = \begin{cases} 0 & \text{if } x < 0 \text{ or } x > 1 \\ \int_0^1 \frac{12}{7}(xy + x^2) \, dy = \frac{6y}{7} + \frac{12x^3}{7} & \text{if } 0 \leq x \leq 1 \\ \end{cases}
\] 
The marginal density of $Y$ is \[
f_Y(y) = \begin{cases} 0 & \text{if } y < 0 \text{ or } y > 1 \\ \int_0^1 \frac{12}{7}(xy + x^2) \, dx = \frac{1}{7} + \frac{6y}{7} & \text{if } 0 \leq y \leq 1 \\ \end{cases}
\] 
The product of these marginals is not the original joint density, so they are not independent.
6. **GW 26.7**: a) As we will see below, \( f_X(x)f_Y(y) \neq f_{X,Y}(x,y) \), so these random variables are *not* independent.

b) The marginal density of \( X \) is

\[
 f_X(x) = \begin{cases} 
 0 & \text{if } x < 0 \text{ or } x > 10 \\
 \int_0^{10-x} \frac{3xy}{1250} \, dy = \frac{3(x-10)^2}{1250} & \text{if } 0 \leq x \leq 10.
\end{cases}
\]

The marginal density of \( Y \) is

\[
 f_Y(y) = \begin{cases} 
 0 & \text{if } y < 0 \text{ or } y > 10 \\
 \int_0^{10-y} \frac{3xy}{1250} \, dx = \frac{3(y-10)^2}{1250} & \text{if } 0 \leq y \leq 10.
\end{cases}
\]

7. **GW 26.8**: We want \( \Pr(X < Y) \). Assuming independence (otherwise we have no chance to solve the problem), the joint density of \( X \) and \( Y \) is

\[
 f_{X,Y}(x,y) = 120e^{-12x-10y}
\]

(if \( x, y \geq 0 \); it’s 0 otherwise).

\( \Pr(X < Y) \) is the double integral of this density function over the shaded region in figure 4 of the figures page, i.e.

\[
 \Pr(X < Y) = \int_{x=0}^{\infty} \int_{y=x}^{\infty} 120e^{-12x-10y} \, dy \, dx = \frac{6}{11} \approx 0.5454.
\]

8. **GW 26.12**: a) On the triangle (area 8) the joint density is \( \frac{1}{8} \). The probability that \( X < 3, Y < 3 \) is the area of the shaded region in figure 5 of the figures page, divided by 8, i.e. 7/8.

9. **GW 28.1**: The marginal density of \( X \) is

\[
 f_X(x) = \begin{cases} 
 0 & \text{if } x < 0 \text{ or } x > 3 \\
 \int_0^{3-x} \frac{2}{9} \, dy = \frac{2(3-x)}{9} & \text{if } 0 \leq x \leq 3.
\end{cases}
\]

So \( E(X) = \int_0^3 x \frac{2(3-x)}{9} \, dx = 1 \).

10. **GW 28.13**: \( E(X) = \int_0^5 \frac{3x}{245} (x^2 + 8) \, dx = \frac{615}{196} \approx 3.13776 \).

11. **GW 28.21**: The marginal density of \( X \) is

\[
 f_X(x) = \begin{cases} 
 0 & \text{if } x < 0 \text{ or } x > 2 \\
 \int_{y=0}^{4} \frac{3}{80} (x^2 + y) \, dy = \frac{3}{10} + \frac{3x^2}{20} & \text{if } 0 \leq x \leq 2.
\end{cases}
\]

So \( E(X) = \int_0^2 x \left( \frac{3}{10} + \frac{3x^2}{20} \right) \, dx = \frac{6}{5} = 1.2 \).
12. (a) 28.22 with one child:

Let $D$ be the point of the perimeter of the rink where the door is located. Let $D'$ be the point opposite the door. Imagine breaking the perimeter at $D'$, and rolling it out as a straight line on the $x$-axis with $D$ at the origin (so it stretches from $-50\pi$ to $50\pi$). Let $X$ be the position of the child when the mother arrives; $X$ is uniform on $[-50\pi, 50\pi]$, so has density:

$$f_X(x) = \begin{cases} 
0 & \text{if } x < -50\pi \text{ or } x > 50\pi \\
\frac{1}{100\pi} & \text{if } -50\pi \leq x \leq 50\pi.
\end{cases}$$

The mother is interested in the distance from her to the child. This is $Z = |X|$. The CDF of $Z$ is given by

$$F_Z(z) = \begin{cases} 
0 & \text{if } z < 0 \\
\Pr(Z \leq z) = \Pr(-z \leq X \leq z) = \frac{2z}{100\pi} & \text{if } 0 \leq z \leq 50\pi \\
1 & \text{if } z > 50\pi.
\end{cases}$$

The density of $Z$ is given by differentiating the CDF:

$$f_Z(z) = \begin{cases} 
0 & \text{if } z < 0 \text{ or } z > 50\pi \\
\frac{1}{50\pi} & \text{if } 0 \leq z \leq 50\pi.
\end{cases}$$

So $E(Z) = \int_0^{50\pi} \frac{z}{50\pi} \, dz = 25\pi$.

(b) 28.22 with two children:

As before let $D$ be the point of the perimeter of the rink where the door is located. Let $D'$ be the point opposite the door. Imagine breaking the perimeter at $D'$, and rolling it out as a straight line on the $x$-axis with $D$ at the origin (so it stretches from $-50\pi$ to $50\pi$). Let $X$ be the position of the first child when the mother arrives and $Y$ the position of the second; $X, Y$ are uniform on $[-50\pi, 50\pi]$, and independent, so have joint density:

$$f_{X,Y}(x,y) = \begin{cases} 
0 & \text{if } x < -50\pi \text{ or } x > 50\pi \text{ or } y < -50\pi \text{ or } y > 50\pi \\
\frac{1}{(100\pi)^2} & \text{if } -50\pi \leq x, y \leq 50\pi.
\end{cases}$$

The mother is interested in the distance from her to the nearest child. This is $Z = \min\{|X|, |Y|\}$. The CDF of $Z$ is given by

$$F_Z(z) = \begin{cases} 
0 & \text{if } z < 0 \\
\Pr(Z \leq z) = \Pr(\min|X|, |Y| \leq z) = \frac{2z}{100\pi} & \text{if } 0 \leq z \leq 50\pi \\
1 & \text{if } z > 50\pi.
\end{cases}$$

To calculate $\Pr(\min|X|, |Y| \leq z)$, note that the event we are looking at is the union of $|X| \leq z$ and $|Y| \leq z$ (one or the other), and so

$$\Pr(\min|X|, |Y| \leq z) = \Pr(|X| \leq z) + \Pr(|Y| \leq z) - \Pr(|X| \leq z, |Y| \leq z) = \frac{2z}{100\pi} + \frac{2z}{100\pi} - \left(\frac{2z}{100\pi}\right)^2$$
(the very last part because $X, Y$ are independent). So the density of $Z$, which is given by differentiating the CDF, is:

$$f_Z(z) = \begin{cases} 0 & \text{if } z < 0 \text{ or } z > 50\pi \\ \frac{1}{25\pi} - \frac{2z}{(50\pi)^2} & \text{if } 0 \leq z \leq 50\pi. \end{cases}$$

So $E(Z) = \int_0^{50\pi} z \left( \frac{1}{25\pi} - \frac{2z}{(50\pi)^2} \right) dz = (50\pi)/3$.

13. The following density is a special case of one that occurs fairly commonly in economics and social science. It’s called the Pareto or Zipf density (Wikipedia has good pages on both). There’s a Pareto density for each $\alpha > 1$, and it’s given by

$$f_\alpha(x) = \begin{cases} 0 & \text{if } x < 1 \\ \frac{c}{x^\alpha} & \text{if } x \geq 1. \end{cases}$$

Here $c$ is a constant that depends on $\alpha$.

(a) For each $\alpha > 1$, find the value of $c = c(\alpha)$ that makes $f_\alpha$ a valid density:

$$\int_1^\infty \frac{c}{x^\alpha} \, dx = \left[ \frac{c}{(1-\alpha)x^{\alpha-1}} \right]_{x=1}^\infty = \frac{c}{\alpha - 1}.$$ 

So we must take $c = \alpha - 1$.

(b) For which values of $\alpha$ does the density $f_\alpha$ have a finite expectation:

$$E(X) = \int_1^\infty \frac{\alpha - 1}{x^\alpha} \, dx = \int_1^\infty \frac{\alpha - 1}{x^{\alpha-1}} \, dx = \left[ \frac{\alpha - 1}{(2-\alpha)x^{\alpha-2}} \right]_{x=1}^\infty.$$ 

If $\alpha > 2$ then this integral converges (to $\frac{\alpha-1}{\alpha-2}$). If $\alpha \leq 2$ then it diverges. So there is finite expectation for $\alpha > 2$. 

4