Introduction to Probability

Math 30530, Section 01 — Fall 2012

Homework 8 — Solutions

1. **GW** 29.1a): The joint density is 2/9 on the triangle (and 0 elsewhere).

$$E(X+Y) = \int\int_{\text{triangle}} \frac{2}{9}(x+y)\ dA = \int_{x=0}^{3}\int_{y=0}^{3-x} \frac{2}{9}(x+y)\ dydx = 2.$$

2. **GW** 29.4:

$$E(Y) = \int_{x=0}^{\infty}\int_{y=0}^{x} y \times 18e^{-2x-7y}\ dydx = \frac{1}{9};$$

$$E(Y^2) = \int_{x=0}^{\infty}\int_{y=0}^{x} y^2 \times 18e^{-2x-7y}\ dydx = \frac{2}{81};$$

so

$$\mathrm{Var}(Y) = E(Y^2) - (E(Y))^2 = \frac{1}{81}.$$

3. **GW** 29.24:

$$E((x-y)/2) = \int_{x=0}^{4}\int_{y=0}^{3} \left(\frac{x-y}{2}\right)\frac{xy^2}{72}\ dydx = \frac{5}{24}.$$

4. **GW** 31.11: The density of X is on the interval $[5, 30]$, and 0 elsewhere.

a)

$$E(Y) = 2.5E(X) + 3;$$

$$E(X) = \frac{30+5}{2} = 17.5;$$

so

$$E(Y) = 2.5 \times 17.5 + 3 = 46.75.$$

b)

$$\mathrm{Var}(Y) = (2.5)^2\mathrm{Var}(X);$$

$$\mathrm{Var}(X) = \frac{(30-5)^2}{12} = 52.083\ldots;$$

so

$$\mathrm{Var}(Y) = (2.5)^2 \times 52.083\ldots = 325.52083\ldots$$

and the standard deviation is $\sqrt{325.52083} \approx 18.04$.

5. **GW** 32.8:

$$\Pr(\max(X,Y,Z) \le 20) = \Pr(X < 20, Y < 20, Z < 20) = \Pr(X < 20)\Pr(Y < 20)\Pr(Z < 20).$$

The average for X is 20 minutes, so $1/\lambda = 20$, and $\lambda = 1/20$.

$$\Pr(X < 20) = \int_0^{20} \frac{1}{20} e^{-x/20}\,dx = \left[-e^{-x/20}\right]_{x=0}^{20} = 1 - (1/e),$$

with the same for $\Pr(Y < 20)$ and $\Pr(Z < 20)$; so

$$\Pr(\max(X,Y,Z) \le 20) = \left(1 - \frac{1}{e}\right)^3 \approx .2638.$$

6. **GW** 32.14:

$$\Pr(Y > 1/\lambda) = \int_{1/\lambda}^{\infty} \lambda e^{-\lambda x}\,dx = \left[-e^{-\lambda x}\right]_{1/\lambda}^{\infty} = 1/e.$$

7. **GW** 32.17: a)

$$p_Y(y) = \int_y^{y+1} \lambda e^{-\lambda x}\,dx = \left[-e^{-\lambda x}\right]_y^{y+1} = e^{-\lambda y} - e^{-\lambda(y+1)} = \left(e^{-\lambda}\right)^y \left(1 - e^{-\lambda}\right).$$

b) Y is *almost* a geometric random variable, with $p = 1 - e^{-\lambda}$. However, it is not exactly geometric, since it takes values $0, 1, 2, \ldots$ (rather than $1, 2, 3, \ldots$). But $Y + 1$ is geometric with $p = 1 - e^{-\lambda}$.

8. **GW** 35.28: a) If X is height, $X \sim \mathcal{N}(64, (4.8)^2)$, so

$$\Pr(X > 66) = \Pr(Z > 2/(4.8) = .42) = .3372.$$

b) $Y \sim \text{Binomial}(10, .3372)$, so

$$\Pr(Y = 7) = \binom{10}{7}(.3372)^7(.6628)^3 \approx .0173.$$

9. I break a piece of chalk that is 6 inches long into three pieces, by picking two spots uniformly and independently at which to make breaks. What is the probability that the three pieces can be combined to form a triangle?

Solution: Let X and Y be the two spots along the chalk at which I break it. The joint density of X and Y is $1/36$ on the square $[0,6] \times [0,6]$, and 0 elsewhere. We need to identify the part of this square that corresponds to it being possible to make a triangle of of the three pieces, and compute the area of that part (and then divide it by 36).

If $X < Y$ (upper triangular part of the square obtained by adding the line $x = y$) then the lengths of the three pieces are X, $Y - X$ and $6 - Y$. In order for a triangle to be possible, each of these three pieces must be no longer than the sum of the other two (triangle inequality). This gives three equations that need to be satisfied in the upper triangle:

$$X + (Y - X) \geq 6 - Y, \quad X + (6 - Y) \geq Y - X, \quad (Y - X) + (6 - Y) \geq X$$

which reduces to

$$Y \geq 3, \quad Y - X \leq 3, \quad X \leq 3$$

(i.e., all three pieces length at most 1/2). This leads to a viable region that is a triangle with vertices $(0,3),(3,3),(3,6)$, area 9/2.

If $X > Y$ (lower triangular part of the square obtained by adding the line $x = y$) then the same reasoning leads to a viable region that is a triangle with vertices $(3,0),(3,3),(6,3)$, area 9/2.

The total area of the viable region is 9, so the probability that a triangle can be formed is $9/36 = 1/4$.

10. Let X be exponential with parameter λ. We know that $E(X) = 1/\lambda$ and $E(X^2) = 2/\lambda^2$.

    (a) Find $E(X^3)$ (for this, you might think that you have to do integration by parts three times; but after the first integration by parts, you should be able to use the fact that you know that $E(X^2) = 2/\lambda^2$ to finish the problem without any more integration).

    Solution: Using integration by parts with $u = x^3$ and $dv = \lambda e^{-\lambda x}\, dx$, we get

    $$E(X^3) = \int_0^\infty x^3 \lambda e^{-\lambda x}\, dx = \left[-x^3 e^{-\lambda x}\right]_0^\infty + 3\int_0^\infty x^2 e^{-\lambda x}\, dx = 0 + 3\frac{E(X^2)}{\lambda} = \frac{6}{\lambda^3}.$$

    (b) Find $E(X^4)$ (again, one integration by parts should do, together with your result from the last part).

    Solution: Using integration by parts with $u = x^4$ and $dv = \lambda e^{-\lambda x}\, dx$, we get

    $$E(X^4) = \int_0^\infty x^4 \lambda e^{-\lambda x}\, dx = \left[-x^4 e^{-\lambda x}\right]_0^\infty + 4\int_0^\infty x^3 e^{-\lambda x}\, dx = 0 + 4\frac{E(X^3)}{\lambda} = \frac{24}{\lambda^4}.$$

    (c) Guess what is $E(X^n)$, based on the pattern from the previous answers.

    Solution: The pattern seems to be

    $$E(X^n) = \frac{n!}{\lambda^n}.$$

(d) If you are familiar with proof by induction, show that your guess is correct. (If not, just ignore this part.)

Solution: We prove the guess from the last part by induction on n; the base case $n = 1$ is done (it is $E(X) = 1/\lambda$). For $n > 1$,

$$E(X^n) = \int_0^\infty x^n \lambda e^{-\lambda x}\, dx = \left[-x^n e^{-\lambda x}\right]_0^\infty + n\int_0^\infty x^{n-1} e^{-\lambda x}\, dx = 0 + n\frac{E(X^{n-1})}{\lambda} = \frac{n!}{\lambda^n}.$$

11. Let X be a uniform random number between 0 and 1, and let $0 < a < b$ be constants.

(a) Find the distribution of $(b-a)X + a$.

Solution: Let $Z = (b-a)X + a$; the range of values for Z are from a to $(b-a)1 + a = b$, so $F_Z(z)$ is 0 for $z < a$ and 1 for $z > b$. For $a \le z \le b$,

$$F_Z(z) = \Pr(Z \le z) = \Pr(X \le \frac{z-a}{b-a}) = \frac{z-a}{b-a}.$$

(b) Find the density function of $(b-a)X + a$.

Solution: Differentiating $F_Z(z)$ we get the density to be 0 if $z < a$ or $z > b$, and $\frac{1}{b-a}$ if $a \le z \le b$.

(c) Compare your answer to the last part with the density function of a uniform random number between a and b. Are they the same? (**Hint**: yes they are, unless you've made a computation error. In general, if $X \sim \text{Uniform}(0,1)$ then $(b-a)X + a \sim \text{Uniform}(a,b)$.)

Solution: Yes, they are the same.

12. Someone calls the police to tell them that a bomb has been left in a public place, and that it will go off at a random time that is an exponential random variable with expected value 2 hours. The police contact the bomb squad, who tells them that it will take them a quarter of an hour to get to the site of the bomb, and that they can then diffuse it in a time that is an exponential random variable with expected value half an hour.

(a) What is the probability that the bomb will go off before the bomb squad arrives?

Solution: Let X be the time the bomb goes off; $X \sim \text{Exponential}(.5)$ (I'm using hours as units; for $E(X) = 2$, want $\lambda = .5$). We want:

$$\Pr(X < .25) = \int_0^{.25} .5e^{-.5x}\, dx = \left[-e^{-.5x}\right]_0^{.25} = 1 - e^{-.125} \approx .12.$$

(b) The bomb squad arrives, and the bomb has not yet gone off. What is the probability, if the bomb squad does nothing, that it goes off in the next quarter of an hour?

Solution: By memorylessness, it is exactly to same at the answer to the previous part: $\approx .12$.

(c) The moment that the bomb squad has been contacted, a reporter calls the police and asks what is the probability that the bomb will be successfully diffused. What should the police reply?

Solution: Let Y be the time that it takes the bomb squad to diffuse the bomb, once they have arrived; $Y \sim$ Exponential(4) (again using hours as units; for $E(Y) = 1/4$, want $\lambda = 4$). We want the probability that $Y + .25 < X$ ($Y + .25$ is the total time it takes to diffuse the bomb, from the moment the call is made). Using the product of the densities of X and Y as their joint density (assuming independence), we have

$$\Pr(Y + .25 < X) = \int_{x=.25}^{\infty} \int_{y=0}^{x-.25} 2e^{-.5x-4y}\, dydx \approx .784.$$

13. A coffee dispenser releases into a 10 ounce cup an amount of coffee that is normally distributed with mean μ ounces (this value can be set by the operator by turning a dial) and standard deviation .5 ounces.

(a) If μ is set to 9.5, what is the probability that the coffee will overflow the cup?

Solution: Let X be the amount of coffee dispensed; $X \sim \mathcal{N}(9.5, (.5)^2)$. We want

$$\Pr(X > 10) = \Pr(Z > 1) \approx .1587.$$

(b) Customers are unhappy when their cups have fewer than 8.8 ounces of coffee in them. What proportion of customers are unhappy when μ is set to 9.5?

Solution: We want

$$\Pr(X < 8.8) = \Pr(Z < -1.4) \approx .0808.$$

(c) What value should the operator set μ to be, to be sure that only 5% of cups overflow?

Solution: Now $X \sim \mathcal{N}(\mu, (.5)^2)$. We want

$$\Pr(X < 10) = 95.$$

This is the same as

$$\Pr(Z < \frac{10-\mu}{.5}) = .95.$$

Since $\Pr(Z < 1.645) = .95$, we need to solve $(10-\mu)/.5 = 1.645$, or $\mu = 9.1775$.

(d) One day the operator discovers that the "μ" dial is jammed at $\mu = 9.5$. He can adjust σ by altering the tightness of a screw on the back of the dispenser. What value of σ should he select so that only 5% of cups overflow?

Solution: Now $X \sim \mathcal{N}(9.5, \sigma^2)$. We want

$$\Pr(X < 10) = 95.$$

This is the same as

$$\Pr(Z < \frac{.5}{\sigma}) = .95.$$

Since $\Pr(Z < 1.645) = .95$, we need to solve $.5/\sigma = 1.645$, or $\sigma \approx .304$.