Puzzler 1

The puzzle:

I have two coins in my pocket. One is fair: it comes up heads half the time. The other is biased in favour of heads: it comes up heads with probability .7 each time I toss it. I can’t tell the two coins apart.

I pick one coin from my pocket. Clearly the probability that it is the fair coin is .5. But then, I toss it four times in a row, and each time I get a head. Clearly this should change my assessment of the probability that I am holding the fair coin - it should make that probability somewhat less than .5. But how much less?

So here’s the puzzle: what’s the probability that I picked the fair coin, given that when I tossed it four times in a row, it came up heads each time?

Solution:

Consider the experiment of picking a coin out of my pocket, recording whether it is fair or biased, then tossing it four times in a row, and recording the outcome of each toss.

Let $F$ be the event that I picked the fair coin, $B$ be the event that I picked the biased coin, and $4H$ the event that I saw 4 heads in a row. By Bayes’ formula,

$$
Pr(F|4H) = \frac{Pr(F \cap 4H)}{Pr(4H)} = \frac{Pr(4H|F)Pr(F)}{Pr(4H|F)Pr(F) + Pr(4H|B)Pr(B)} = \frac{(0.5)^4 \times 0.5}{(0.5)^4 \times 0.5 + (0.7)^4 \times 0.5} \approx 0.2065.
$$

More of the puzzle:

And what’s the answer when I replace “four” with “$n$”, for an arbitrary whole number $n$?

Solution:

Change the experiment by observing $n$ tosses in a row, rather than just 4, and let $nH$ be the event that I saw $n$ heads in a row. Bayes’ formula now gives

$$
Pr(F|nH) = \frac{Pr(nH|F)Pr(F)}{Pr(nH|F)Pr(F) + Pr(nH|B)Pr(B)} = \frac{(0.5)^n}{(0.5)^n + (0.7)^n}.
$$

Notice that the .5’s for $Pr(F), Pr(B)$ cancel. Notice also that

$$
Pr(F|0H) = 0.5 \text{ and } \lim_{n \to \infty} Pr(F|nH) = 0;
$$

both are exactly what we should expect.
Correct solvers:

- Kevin Katalinic (A♥, 2♥)
- Connor Voglewede (3♥, 4♥)
- John Macke (5♥, 6♥)
- Roisin McCord (7♥, 8♥, winner!)
- Yutong Zhang (partial) (9♥)
- Liz Quinn (10♥, J♥)
- Grace Smith (Q♥, K♥)
- Hannah Pawelczyk (A♠, 2♠)
- Melissa Flynn (partial) (3♠)
- Michael Fronk (4♠, 5♠)
- Michael MacGillivray (6♠, 7♠)
- Sara Mykrantz (8♠, 9♠)
- Eric Krakowiak (10♠, J♠)
- John Brahier (Q♠, K♠)
- Matt Cole (A♦, 1♦)