Math 30530 — Introduction to Probability

Puzzler 1

Solutions

The puzzle:

I have two coins in my pocket. One is *fair*: it comes up heads half the time. The other is biased in favour of heads: it comes up heads with probability .7 each time I toss it. I can't tell the two coins apart.

I pick one coin from my pocket. Clearly the probability that it is the fair coin is .5. But then, I toss it four times in a row, and each time I get a head. Clearly this should change my assessment of the probability that I am holding the fair coin - it should make that probability somewhat less than .5. But how much less?

So here's the puzzle: what's the probability that I picked the fair coin, given that when I tossed it four times in a row, it came up heads each time?

Solution:

Consider the experiment of picking a coin out of my pocket, recording whether it is fair or biased, then tossing it four times in a row, and recording the outcome of each toss.

Let F be the event that I picked the fair coin, B be the event that I picked the biased coin, and $4H$ the event that I saw 4 heads in a row. By Bayes' formula,

$$\Pr(F|4H) = \frac{\Pr(F \cap 4H)}{\Pr(4H)} = \frac{\Pr(4H|F)\Pr(F)}{\Pr(4H|F)\Pr(F) + \Pr(4H|B)\Pr(B)} = \frac{(.5)^4 \times .5}{(.5)^4 \times .5 + (.7)^4 \times .5} \approx .2065.$$

More of the puzzle:

And what's the answer when I replace "four" with "n", for an arbitrary whole number n?

Solution:

Change the experiment by observing n tosses in a row, rather than just 4, and let nH be the event that I saw n heads in a row. Bayes' formula now gives

$$\Pr(F|nH) = \frac{\Pr(nH|F)\Pr(F)}{\Pr(nH|F)\Pr(F) + \Pr(nH|B)\Pr(B)} = \frac{(.5)^n}{(.5)^n + (.7)^n}.$$

Notice that the .5's for $\Pr(F)$, $\Pr(B)$ cancel. Notice also that

$$\Pr(F|0H) = .5 \quad \text{and} \quad \lim_{n\to\infty} \Pr(F|nH) = 0;$$

both are exactly what we should expect.

Correct solvers:

- Kevin Katalinic ($A\heartsuit, 2\heartsuit$)
- Connor Voglewede ($3\heartsuit, 4\heartsuit$)
- John Macke ($5\heartsuit, 6\heartsuit$)
- Roisin McCord ($7\heartsuit, 8\heartsuit$, winner!)
- Yutong Zhang (partial) ($9\heartsuit$)
- Liz Quinn ($10\heartsuit, J\heartsuit$)
- Grace Smith ($Q\heartsuit, K\heartsuit$)
- Hannah Pawelczyk ($A\spadesuit, 2\spadesuit$)
- Melissa Flynn (partial) ($3\spadesuit$)
- Michael Fronk ($4\spadesuit, 5\spadesuit$)
- Michael MacGillivray ($6\spadesuit, 7\spadesuit$)
- Sara Mykrantz ($8\spadesuit, 9\spadesuit$)
- Eric Krakowiak ($10\spadesuit, J\spadesuit$)
- John Brahier ($Q\spadesuit, K\spadesuit$)
- Matt Cole ($A\diamondsuit, 1\diamondsuit$)