Math 30530 — Introduction to Probability

Puzzler 2

Solutions

The puzzle:

Ann, Brian and Cora hold a three-person paintball duel. The rules are as follows: Each stands at one corner of a triangle. Initially, Ann chooses a target (either Brian or Cora) and shoots. If she has a hit, the hit target leaves the game, otherwise both stay. The paint gun then moves to Brian (if he's still in the game), who repeats the process, and then hands the gun to Cora (if she's still in the game). This cyclic rotation of the paint gun among (whoever is still left in the game of) Ann, Brian, Cora, Ann, Brian, Cora, ... goes on until only only person is left unhit; that person is the winner.

Ann hits her targets 30% of the time, independently from shot to shot; Brian hits his target 100% of the time; and Cora hits hers half the time.

There's one extra rule: initially, Ann has the option of not shooting, i.e., of allowing Brian the first shot.

Here's the puzzle: What strategy does Ann adopt to maximize her chances of winning the duel? Assume that all other players are also attempting to maximize their chances of winning, and no player holds a grudge: choices of who to shoot at are made purely with an eye to maximizing chances of winning, and never to help or hinder another player.

Solution:

I'll do the calculations for Ann having a hit probability a, and Cora a hit probability c, for some $0 < a < c < 1$, and then specialize at the end.

Suppose Ann chooses to fire first. She can choose either Brian or Cora as her target. If she chooses Cora, then with probability a she has a hit, and so losses the game: Brian will certainly hit Ann on his first shot. If she misses Cora (a $1 - a$ probability), then it's exactly the same as if she choose to pass, so she wins with probability p, where p is the probability that she wins if Brian shoots first in the three person game. So overall, her probability of winning if she chooses to aim first at Cora is $(1 - a)p$.

(What is p? Brian has to choose to aim at (and hit) either Ann or Cora. If he aims and hits Ann, then he has a probability $1 - c$ of winning: exactly the probability that Cora misses him. If he aims and hits Cora, then he has a probability $1 - a$ of winning: exactly the probability that Ann misses him. Since $a < c$, we have $1 - a > 1 - c$, so Brian will choose to aim at Cora first. In this case, Ann's probability of winning is a; so $p = a$.)

If Ann chooses to aim first at Brian, and misses (a $1 - a$ probability), then as before she subsequently wins with probability p. If she hits Brian (probability a), then she wins with probability q, where q is the probability that she wins a 2-person duel with Cora, where Cora aims first. So overall, her probability of winning if she chooses to aim first at Brian is $(1 - a)p + aq$. Clearly, this is greater than $(1 - a)p$, so we can conclude the following already: if Ann chooses not to pass, she will first aim at Brian, and wins with probability $(1 - a)p + aq$.

(What is q? We have to consider infinitely many possibilities for Ann winning a 2-person duel with Cora: Cora misses, Ann hits; Cora misses, Ann misses, Cora misses, Ann hits; etc. The net probability is

$$(1-c)a + (1-c)(1-a)(1-c)a + (1-c)(1-a)(1-c)(1-a)(1-c)a + \dots.$$

This is a geometric series, starting at $(1-c)a$ with common ratio $(1-c)(1-a)$, so the sum is

$$q = \frac{(1-c)a}{1-(1-c)(1-a)}.)$$

Now suppose Ann chooses to pass. Then she wins with probability $p = a$ (this is exactly the situation we set up p for).

Now we come to the computation. Which is bigger:

$$(1-a)p + aq \ \left(= (1-a)a + \frac{(1-c)a^2}{1-(1-c)(1-a)}\right) \qquad \text{or} \qquad p \ (= a)?$$

For our specific values of $a = .3$, $b = .5$, these two numbers become $.27923\ldots$ and $.3$. The latter is larger, and so we conclude:

Ann's best strategy is to initially pass.

More generally, if Ann has probability a and Cora probability $c > a$ (and Brian probability 1), then it will be in Ann's interest to pass initially, as long as

$$a > (1-a)a + \frac{(1-c)a^2}{1-(1-c)(1-a)}.$$

Rearranging terms, this is the same as

$$a > \frac{1-2c}{1-c}.$$

So, for example, if Ann has a 30% chance of hitting her targets, then she should pass exactly when $.3 > (1-2c)/(1-c)$, which is the same as $c > 7/17 \approx .412$, and she should aim for Brian otherwise.

Fully correct solvers:

- Michael MacGillivray (Ace♡, 2♡)
- Ningzhou Shen (3♡, 4♡)
- Roisin McCord (5♡, 6♡)
- John Macke (7♡, 8♡)
- Matt Cole (9♡, 10♡)

Not quite fully correct solvers:

- Kaitlyn Keily (Jack♡)
- Melissa Flynn (Queen♡, winner!)
- Alex Jarocki (King♡)
- Conor Voglewede (Ace♠)