Math 30530: Introduction to Probability, Fall 2013

Midterm Exam II

Practice exam — solutions

1. I'm taking part in the All-Ireland hay-tossing championship next week (hay-tossing is a real sport in Ireland & Scotland — see e.g. `http://scottishheavyathletics.com/sheaf.html`). The height I can throw a bale of hay (in yards) is a random variable with density function

$$f(x) = \begin{cases} \frac{c}{x^4} & \text{for } x \geq 2 \\ 0 & \text{for } x < 2 \end{cases}$$

   (a) What is c?

   Solution:

$$\int_2^\infty \frac{c}{x^4}\, dx = \left[\frac{-c}{3x^3}\right]_2^\infty = \frac{c}{24}.$$

   Since this integral should equal 1, we must have $c = 24$.

   (b) What is the probability that I throw the bale to a height no more than 3 yards?

   Solution:

$$\Pr(2 \leq X \leq 3) = \int_2^3 \frac{24}{x^4}\, dx = \left[\frac{-8}{x^3}\right]_2^3 = 1 - \frac{8}{27} = \frac{19}{27}.$$

   (c) The prize a contestant receives is $100x^2 + 100x$ euro if he or she tosses the bale x yards. What is my expected prize?

   Solution: We want $E(100X^2 + 100X)$. **NOTE**: this is <u>not</u> the same as $100(E(X))^2 + 100E(X)$ (in general $E(g(X)) \neq g(E(X))$, unless g is linear).

$$E(100X^2 + 100X) = \int_2^\infty \frac{24(100x^2 + 100x)}{x^4}\, dx = 1500.$$

   So my expected prize is 1500 euros.

2. Eggs at certain stall at the South Bend farmers market have a weight that is normally distributed with mean 2oz and standard deviation .2oz. The weights of individual eggs are independent of each other.

   (a) I buy six eggs. What is the probability that at most one of them weighs less than 1.7oz?

   Solution: Let X be the weight of a single egg; $X \sim \mathcal{N}(2, (.2)^2)$.

$$\Pr(X < 1.7) = \Pr(Z < -1.5) = .0668.$$

Now let Y be the number of eggs from among the six that weight less than 1.7oz. By what we've just calculated, $Y \sim \text{Binomial}(6, .0668)$. So the probability we want is

$$\Pr(Y \leq 1) = \binom{6}{0}(.0668)^0(.9332)^6 + \binom{6}{1}(.0668)^1(.9332)^5.$$

(b) The egg vendor weighs one egg and tells me that it is very light; so light, in fact, that only 3% of all his eggs are that light or lighter. What's the weight of the egg?

Solution: We want to find that x such that $\Pr(X < x) = .03$, where $X \sim \mathcal{N}(2, (.2)^2)$. Standardizing, this is the same as $\Pr(Z < (x-2)/.2) = .03$. From a standard normal table, $\Pr(Z < -1.88) = .03$, so we want x such that $(x-2)/.2 = -1.88$. So $x = 1.624$.

(c) A competing vendor, whose eggs also have a weight that is normally distributed with mean 2oz, promises me that she can be 95% confident that one of her eggs weights 7.8 and 8.2 oz. What is the standard deviation of the weights of her eggs?

Solution: Suppose std. dev. is σ (so if X is weight, $X \sim \text{Normal}(2, \sigma)$). Know:

$$\Pr(7.8 < X < 8.2) = .95,$$

so, normalizing,

$$\Pr(-.2/\sigma < Z < .2/\sigma) = .95,$$

where Z is standard normal. From table,

$$\Pr(-1.96 < Z < 1.96) = .95,$$

so $.2/\sigma = 1.96$, $\sigma \approx .102$.

3. A pair of random variables X, Y have joint density $f(x,y)$ that takes value $\frac{3}{4}x$ in the triangle with vertices $(0,0)$, $(2,0)$ and $(0,2)$, and value 0 elsewhere.

(a) Find the marginal density of X.

Solution:

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\, dy = \begin{cases} 0 & \text{if } x < 0 \text{ or } x > 2 \\ \int_0^{2-x} \frac{3}{4}x\, dy = \frac{3}{4}x(2-x) & \text{if } 0 \leq x \leq 2. \end{cases}$$

(b) Find the distribution function (CDF) of $X + Y$.

Solution: If $z < 0$, $\Pr(X + Y < z) = 0$ trivially; if $z > 2$, $\Pr(X + Y < z) = 1$ trivially. If $0 \leq z \leq 2$, then

$$\Pr(X + Y \leq z) = \int_0^z \int_0^{z-x} \frac{3}{4}x\, dydx = z^3/8.$$

So

$$F_{X+Y}(z) = \begin{cases} 0 & \text{if } z < 0 \\ z^3/8 & \text{if } 0 \leq z \leq 2 \\ 1 & \text{if } z > 2. \end{cases}$$

(c) Find the density of $X + Y$

Solution:

$$f_{X+Y}(z) = \frac{d}{dz} F_{X+Y}(z) = \begin{cases} 0 & \text{if } z < 0 \\ 3z^2/8 & \text{if } 0 \leq z \leq 2 \\ 0 & \text{if } z > 2. \end{cases}$$

4. My dog Casey (`http://www3.nd.edu/~dgalvin1/personal/Casey/Casey1.jpg`) spends most of her time staring out the window watching for squirrels. They run by at random times, at a constant average of one every 10 minutes.

(a) What is the probability that Casey sees two or fewer squirrels in next 30 minutes?

Solution: Let X be the number of squirrels she sees in 30 minutes. A suitable random variable to model X is a Poisson with $\lambda = 3$ (average number of squirrels per 30 minutes). We want:

$$\Pr(X \leq 2) = e^{-3} + 3e^{-3} + \frac{3^2}{2!}e^{-3}.$$

(b) Casey arrives at the window at 2pm. What is the probability that she sees her first squirrel sometime between 2.15pm and 2.20pm?

Solution: Let Y be the time at which she sees here first squirrel. Using minutes as units, a suitable random variable to model Y is an exponential with $\lambda = 1/10$ (average number of squirrels per 1 minutes). We want:

$$\Pr(15 \leq Y \leq 20) = \int_{15}^{20} (1/10)e^{-x/10}\, dx \approx .0877.$$

NOTE: This is <u>not</u> the same as $\Pr(0 \leq Y \leq 5) = \int_0^5 (1/10)e^{-x/10}\, dx$. We are not giving the information hat 15 minutes have passed without a squirrel arriving, so we cannot use memorylessness here.

(c) At 3pm Casey settles down for a nap. The amount of time she naps for is exponential with average 20 minutes. Write down (but don't evaluate) an integral whose value is the probability that no squirrel runs by the window during her nap.

Solution: Let X be the nap time, and Y the time unit the first squirrel arrives after the nap starts. Using minutes as units, $X \sim$ exponential(1/20) (its expectation is 20, so $\lambda = 1/20$), and $Y \sim$ exponential(1/10). The joint density of X and Y is the product of the individual densities (assuming independence). We want:

$$\Pr(X < Y) = \int_0^\infty \int_x^\infty \frac{e^{-x/20 - y/10}}{200}\, dy dx.$$

5. The two parts of this question are unrelated.

(a) The gaps between consecutive clicks of a Geiger counter are (independent) exponential random variables, always with the same parameter. An operator reports that 50% of all gaps are 6 seconds or longer. What is the parameter λ of the gap between consecutive clicks?

Solution: Let X be the length of a gap; $X \sim$ exponential(λ). We know $\Pr(X > 6) = .5$, i.e.

$$\int_6^\infty \lambda e^{-\lambda x}\, dx = .5.$$

This is the same as

$$1 - e^{-6\lambda} = .5,$$

or $\lambda = (\log 2)/6 = .1155\ldots$.

(b) Historical data indicates that the daytime high temperature in South Bend on Christmas day is normally distributed with mean -5 degrees Celsius and standard deviation of 12 degrees Celsius. What is the mean and standard deviation of the temperature measured in degrees Fahrenheit? (To convert from Celsius to Fahrenheit, divide by 5, multiply by 9 and add 32.)

Solution: Let C be the temperature is Celsius; $C \sim \mathbb{N}(-5, 144)$. We want the find the mean and standard deviation of $(9/5)C + 32$. Using $E(aX + b) = aE(X) + b$ and $\mathrm{Var}(aX + B) = a^2\mathrm{Var}(X)$ (valid for *any* random variables), we get

$$E((9/5)C + 32) = (9/5)(-5) + 32 = 23$$

and

$$\mathrm{Var}((9/5)C + 32) = (9/5)^2(144).$$

So the mean is 23 and the standard deviation is $\sqrt{(9/5)^2(144)} = 21.3$.

6. Let Θ be a random variable that is uniformly distributed on the interval $(-\pi/2, \pi/2)$, and let $Y = \sin\Theta$ (so Y is the y-coordinate of a randomly chosen point on the top half of a unit circle).

(a) What is the distribution function (CDF) of Θ?

Solution:

$$F_\Theta(t) = \begin{cases} 0 & \text{if } t < -\pi/2 \\ \frac{t+\pi/2}{\pi} & \text{if } -\pi/2 \le t \le \pi/2 \\ 1 & \text{if } t > \pi/2. \end{cases}$$

(b) What is the distribution function (CDF) of Y?

Solution: Range of values for Y: -1 to 1. For $-1 \le y \le 1$,

$$\begin{aligned} \Pr(Y \le y) &= \Pr(\sin\Theta \le y) \\ &= \Pr(\Theta \le \sin^{-1} y) \\ &= F_\Theta(\sin^{-1} y) \\ &= \frac{\sin^{-1} y + \pi/2}{\pi}. \end{aligned}$$

So here's the distribution function:

$$F_Y(y) = \begin{cases} 0 & \text{if } y < -1 \\ \frac{\sin^{-1} y+\pi/2}{\pi} & \text{if } -1 \le y \le 1 \\ 1 & \text{if } t > 1. \end{cases}$$

(c) What is the density function of Y?

Solution: Differentiating the distribution function:

$$f_Y(y) = \begin{cases} 0 & \text{if } y < -1 \\ \frac{d}{dy}\left(\frac{\sin^{-1} y + \pi/2}{\pi}\right) = \frac{1}{\pi\sqrt{1-y^2}} & \text{if } -1 \leq y \leq 1 \\ 0 & \text{if } y > 1. \end{cases}$$