1. (a) Let $X$ be a uniformly selected random number on the interval $[0, 1]$. For $a > 0$ and $b \in \mathbb{R}$, let $Y = aX + b$. Calculate the density function of $Y$.

**Solution:** Possible values for $Y$: anything from $b$ to $a + b$. For each $b \leq y \leq a + b$,

$$\Pr(Y \leq y) = \Pr(aX + b \leq y) = \Pr(X \leq (y - b)/a) = (y - b)/a,$$

the last equality using the fact that $X$ is uniform on $[0, 1]$. So, the CDF of $Y$ is

$$F_Y(y) = \begin{cases} 
0 & \text{if } y < a, \\
(y - b)/a & \text{if } a \leq y \leq b + a, \\
1 & \text{if } y > b + a,
\end{cases}$$

and the density of $Y$ is

$$f_Y(y) = \begin{cases} 
0 & \text{if } y < a, \\
1/a & \text{if } a \leq y \leq b + a, \\
0 & \text{if } y > b + a.
\end{cases}$$

(b) Write down the density function of a uniformly selected random number on the interval $[b, a + b]$ ($a > 0$ and $b \in \mathbb{R}$).

**Solution:** Exactly the same as the density of $Y$ in the last part: if $X$ is a uniformly selected random number on the interval $[b, a + b]$, then the density of $X$ is

$$f_X(x) = \begin{cases} 
0 & \text{if } x < a, \\
1/a & \text{if } a \leq x \leq b + a, \\
0 & \text{if } x > b + a.
\end{cases}$$

2. I throw a dart $n$ times at a dartboard with radius 1, each time selecting a uniform and independent point from the board. Let $X_i$ be the random variable that records the distance from my $i$th throw to the center of the dartboard, and let $Y(n)$ be the distance to the center of the dartboard of my closest throw (i.e. $Y(n) = \min\{X_1, \ldots, X_n\}$).
(a) Find the density function of $Y(n)$.

**Solution:** We start with the CDF of $Y(n)$. The possible values of $Y(n)$ are anything from 0 to 1, so we each $0 \leq y \leq 1$ we want to compute $\Pr(Y(n) \leq y)$. It’s easier to compute $\Pr(Y(n) \geq y)$, because

$$
\Pr(Y(n) \geq y) = \Pr(\min\{X_1, \ldots, X_n\} \geq y) = \Pr(X_1 \geq y \text{ AND } X_2 \geq y \text{ AND } \ldots \text{ AND } X_n \geq y) = (\Pr(X_1 \geq y))^n.
$$

The probability that $X_1 \geq y$ is the probability of landing in the annulus outside the circle of radius $y$, which is $(\pi 1^2 - \pi y^2)/(\pi 1^2) = 1 - y^2$. So

$$
\Pr(Y(n) \leq y) = 1 - \Pr(Y(n) \geq y) = 1 - (1 - y^2)^n.
$$

It follows that the CDF of $Y(n)$ is

$$
F_n(y) = \begin{cases} 
0 & \text{if } y < 0, \\
1 - (1 - y^2)^n & \text{if } 0 \leq y \leq 1, \\
1 & \text{if } y > 1,
\end{cases}
$$

and the density of $Y(n)$ is

$$
f_n(y) = \begin{cases} 
0 & \text{if } y < 0, \\
2yn(1 - y^2)^{n-1} & \text{if } 0 \leq y \leq 1, \\
0 & \text{if } y > 1.
\end{cases}
$$

(b) For $n = 1, 2, 3, 4$, find $E(Y(n))$.

**Solution:** From the density, $E(Y(n)) = \int_0^1 2y^2n(1 - y^2)^{n-1} \, dy$. Calculating this integral for $n = 1, 2, 3, 4$ gives values of $2/3$, $8/15$, $16/35$, and $128/315$ (roughly .67, .53, .46, and .41).

(c) For $n = 1, 2, 3, 4$, find $\Pr(Y(n) < .5)$.

**Solution:** From the density, $\Pr(Y(n) < .5) = \int_0^{.5} 2yn(1 - y^2)^{n-1} \, dy$. Calculating this integral for $n = 1, 2, 3, 4$ gives values of $.25$, $.4375$, $.578125$, and $.68359375$.

3. Use the transform of the exponential random variable (which we calculated in class) to compute $E(X)$ and $\text{Var}(X)$ when $X \sim \text{exponential}(\lambda)$.

**Solution:** We computed $M_X(s) = \frac{\lambda}{\lambda - s}$ for $X \sim \text{exponential}(\lambda)$ (as long as $s < \lambda$), so

$$
M'_X(s) = \frac{\lambda}{(\lambda - s)^2}, \quad \text{so} \quad E(X) = M'_X(0) = \frac{1}{\lambda},
$$

and

$$
M''_X(s) = \frac{2\lambda}{(\lambda - s)^3}, \quad \text{so} \quad E(X^2) = M''_X(0) = \frac{2}{\lambda^2},
$$

so

$$
\text{Var}(X) = \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2 = \frac{1}{\lambda^2}.
$$
4. (a) Let $X \sim \text{Poisson}(\lambda)$. Calculate the transform $X$.

**Solution:** $P(K = k) = \frac{\lambda^k}{k!} e^{-\lambda}$ for $X \sim \text{Poisson}(\lambda)$ ($k = 0, 1, 2, 3, \ldots$), so

$$
M_X(s) = E(e^{sX}) = \sum_{k=0}^{\infty} e^{sk} \lambda^k \frac{1}{k!} e^{-\lambda} = e^{-\lambda} \sum_{k=0}^{\infty} \frac{(e^s \lambda)^k}{k!} = e^{-\lambda} e^{e^s \lambda} = e^{\lambda(e^s-1)},
$$

the second-from-last inequality using the power series for $e^x$, $e^x = 1 + x + \ldots + x^{r-1} + \ldots$. This transform is valid for all $s$.

(b) Let $X \sim \text{Poisson}(\lambda_1)$ and $Y \sim \text{Poisson}(\lambda_2)$ be independent. Use transforms to show that $X + Y \sim \text{Poisson}(\lambda_1 + \lambda_2)$.

**Solution:** From previous part,

$$
M_X(s) = e^{\lambda_1(e^s-1)}, \quad M_Y(s) = e^{\lambda_2(e^s-1)}.
$$

Since we know that the transform of a sum of independent rvs is a product of the transforms, we get

$$
M_{X+Y}(s) = e^{\lambda_1(e^s-1)} e^{\lambda_2(e^s-1)} = e^{\lambda_1(e^s-1) + \lambda_2(e^s-1)} = e^{(\lambda_1+\lambda_2)(e^s-1)}.
$$

This is exactly the transform of a Poisson random variable with parameter $\lambda_1 + \lambda_2$, and so we are done.

5. I choose $r$ items from a collection of $N + M$ items, one after the other, without replacement. $N$ of the items are “good” and the remaining $M$ are “bad”. Let $X_i$ be the indicator random variable indicating whether the $i$th item I chose was good (so $X_i = 1$ if the $i$th item was good, and $X_i = 0$ if it was bad). For $i \neq j$, calculate the covariance $\text{Cov}(X_i, X_j)$, and the correlation coefficient. (Note: it should be very small, going to 0 as $N$ and $M$ go to infinity; this justifies treating samples without replacement as being essentially independent when the population is large).

**Solution:** There are $N \cdot \text{times} (N + M - 1)(N + M - 2) \ldots (N + M - (r - 1))$ ways of choosing the $r$ items so that the $i$th is good (think about choosing the $i$th item first, from among the $N$ good items, and choosing the remaining $r - 1$ items arbitrarily from what’s left). There are $(N + M)(N + M - 1)(N + M - 2) \ldots (N + M - (r - 1))$ ways of choosing the $r$ items, in total. So

$$
\Pr(X_i = 1) = \frac{N \times (N + M - 1)(N + M - 2) \ldots (N + M - (r - 1))}{(N + M)(N + M - 1)(N + M - 2) \ldots (N + M - (r - 1))} = \frac{N}{N + M},
$$

and $\Pr(X_i = 0) = M/(N + M)$. Similarly,

$$
\Pr(X_j = 1) = \frac{N}{N + M},
$$
and \( \Pr(X_j = 0) = M/(N + M) \). We thus get that

\[
E(X_i) = \frac{N}{N + M}, \quad E(X_i^2) = \frac{N}{N + M}, \quad \text{Var}(X_i) = \frac{MN}{(N + M)^2},
\]

and

\[
E(X_j) = \frac{N}{N + M}, \quad E(X_j^2) = \frac{N}{N + M}, \quad \text{Var}(X_j) = \frac{MN}{(N + M)^2}.
\]

There are \( N(N-1) \) times \((N+M-2)(N+M-3) \ldots (N+M-(r-1)) \) ways of choosing the \( r \) items so that the \( i \)th and \( j \)th are both good (think about choosing the \( i \)th item first, then the \( j \)th item, from among the \( N \) good items, and choosing the remaining \( r-2 \) items arbitrarily from what’s left). There are \((N+M)(N+M-1)(N+M-2) \ldots (N+M-(r-1)) \) ways of choosing the \( r \) items, in total. So

\[
\Pr(X_i X_j = 1) = E(X_i X_j)
\]

\[
= \frac{N(N-1) \times (N+M-2) \ldots (N+M-(r-1))}{(N+M)(N+M-1)(N+M-2) \ldots (N+M-(r-1))}
\]

\[
= \frac{N(N-1)}{(N+M)(N+M-1)},
\]

and so

\[
\text{Cov}(X_i, X_j) = E(X_i X_j) - E(X_i)E(X_j)
\]

\[
= \frac{N(N-1)}{(N+M)(N+M-1)} - \left( \frac{N}{N+M} \right)^2
\]

\[
= -\frac{NM}{(N+M)^2(N+M-1)}
\]

and

\[
\rho = \frac{\text{Cov}(X_i, X_j)}{\sqrt{\text{Var}(X_i) \text{Var}(X_j)}}
\]

\[
= \frac{-NM}{(N+M)^2(N+M-1)} \cdot \frac{MN}{(N+M)^2}
\]

\[
= -\frac{1}{N + M - 1}
\]

This is indeed small; this shows that although \( X_i \) and \( X_j \) are very slightly (negatively) correlated, when \( N + M \) is large they are essentially uncorrelated.

6. Chapter 4, problems 29 and 30 — see the supplementary solution file 1.

7. Chapter 4, problems 17, 18 and 19 — see the supplementary solution file 2.