Introduction to Probability, Fall 2013

Math 30530 Section 01

Homework 3 — solutions

1. Chapter 1, Problem 30
2. Chapter 1, Problem 31
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10. Chapter 1, Problem 56
11. Chapter 1, Problem 57
12. Chapter 1, Problem 58
13. Chapter 1, Problem 60

    Solutions: See the supplementary solutions file on the website.

14. Show that $\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$.

    Solution: In how many ways can you select n balls from a bag that has n red balls and n green balls? Just directly selecting the balls, you get a count of $\binom{2n}{n}$, the right-hand side.

    A more laborious way to do the job is to first decide on some number k between 0 and n, then select k red balls and $n-k$ green balls. This gives a count of

$$\sum_{k=0}^{n} \binom{n}{k}\binom{n}{n-k} = \sum_{k=0}^{n} \binom{n}{k}^2$$

(the equality using $\binom{n}{k} = \binom{n}{n-k}$), which is the left-hand side.

So, since both sides are counting the same thing, they are equal!

15. Calculate the probability that when you roll six ordinary dice, the total number of different numbers that come up is exactly four.

    Solution: Ω here is all the different ways of rolling six dice, so $|\Omega| = 6^6$ (I'm thinking of the dice being rolled in a particular order).

    Let A be the event that exactly four different numbers come up. We want to find $|A|$.

    One way to get four different numbers is to have three of one, and one each of the other 3. There are 6 ways to decide which is the number that comes up 3 times, and $\binom{5}{3} = 10$ ways to decide which are the numbers that come up once. So there are 60 choices for the "type" of the roll, for example three 1's, one 2, one 3, one 4. To count how many ways we can get this "type" coming up, we have to split the six dice into four classes, with the first class of size three (selecting the three dice that come up 1), the second class of size one (selecting the one dice that comes up 2), the third class of size one (selecting the one dice that comes up 3), and the fourth class of size one (selecting the one dice that comes up 2). The number of ways to do this is $\binom{6}{3,1,1,1} = 120$. So there are 7200 different rolls with three of one kind, and one of each of three other kinds.

    The only other way to get four different numbers is to have two of one, two of another, and one each of the other 2. As in the last paragraph, we get that the total number of ways for this to happen is

    $$\binom{6}{2} \times \binom{4}{2} \binom{6}{2,2,1,1} = 16200$$

    (choose the two numbers that each appear twice, choose the two numbers that each appear once, split the dice into classes of sizes 2,2,1 and 1 to decide which dice show which numbers).

    So $|A| = 7200 + 16200 = 23400$, and

    $$\Pr(A) = \frac{23400}{6^6} = .5015\ldots.$$

    So just a touch over 50%!