MATH 30530 FALL 2013 H3 SOLUTION

Solution to Problem 1.30. Consider the sample space for the hunter's strategy. The events that lead to the correct path are:

(1) Both dogs agree on the correct path (probability p^2, by independence).

(2) The dogs disagree, dog 1 chooses the correct path, and hunter follows dog 1 [probability $p(1-p)/2$].

(3) The dogs disagree, dog 2 chooses the correct path, and hunter follows dog 2 [probability $p(1-p)/2$].

The above events are disjoint, so we can add the probabilities to find that under the hunter's strategy, the probability that he chooses the correct path is

$$p^2 + \frac{1}{2}p(1-p) + \frac{1}{2}p(1-p) = p.$$

On the other hand, if the hunter lets one dog choose the path, this dog will also choose the correct path with probability p. Thus, the two strategies are equally effective.

Solution to Problem 1.31. (a) Let A be the event that a 0 is transmitted. Using the total probability theorem, the desired probability is

$$\mathbf{P}(A)(1-\epsilon_0) + \big(1-\mathbf{P}(A)\big)(1-\epsilon_1) = p(1-\epsilon_0) + (1-p)(1-\epsilon_1).$$

(b) By independence, the probability that the string 1011 is received correctly is

$$(1-\epsilon_0)(1-\epsilon_1)^3.$$

(c) In order for a 0 to be decoded correctly, the received string must be 000, 001, 010, or 100. Given that the string transmitted was 000, the probability of receiving 000 is $(1-\epsilon_0)^3$, and the probability of each of the strings 001, 010, and 100 is $\epsilon_0(1-\epsilon_0)^2$. Thus, the probability of correct decoding is

$$3\epsilon_0(1-\epsilon_0)^2 + (1-\epsilon_0)^3.$$

(d) When the symbol is 0, the probabilities of correct decoding with and without the scheme of part (c) are $3\epsilon_0(1-\epsilon_0)^2 + (1-\epsilon_0)^3$ and $1-\epsilon_0$, respectively. Thus, the probability is improved with the scheme of part (c) if

$$3\epsilon_0(1-\epsilon_0)^2 + (1-\epsilon_0)^3 > (1-\epsilon_0),$$

or

$$(1-\epsilon_0)(1+2\epsilon_0) > 1,$$

which is equivalent to $\epsilon_0 < 1/2$.

(e) Using Bayes' rule, we have

$$\mathbf{P}(0\,|\,101) = \frac{\mathbf{P}(0)\mathbf{P}(101\,|\,0)}{\mathbf{P}(0)\mathbf{P}(101\,|\,0) + \mathbf{P}(1)\mathbf{P}(101\,|\,1)}.$$

The probabilities needed in the above formula are

$$\mathbf{P}(0) = p, \quad \mathbf{P}(1) = 1-p, \quad \mathbf{P}(101\,|\,0) = \epsilon_0^2(1-\epsilon_0), \quad \mathbf{P}(101\,|\,1) = \epsilon_1(1-\epsilon_1)^2.$$

Solution to Problem 1.33. Flip the coin twice. If the outcome is heads-tails, choose the opera. if the outcome is tails-heads, choose the movies. Otherwise, repeat the process, until a decision can be made. Let A_k be the event that a decision was made at the kth round. Conditional on the event A_k, the two choices are equally likely, and we have

$$\mathbf{P}(\text{opera}) = \sum_{k=1}^{\infty} \mathbf{P}(\text{opera} \mid A_k)\mathbf{P}(A_k) = \sum_{k=1}^{\infty} \frac{1}{2}\mathbf{P}(A_k) = \frac{1}{2}.$$

We have used here the property $\sum_{k=0}^{\infty} \mathbf{P}(A_k) = 1$, which is true as long as $\mathbf{P}(\text{heads}) > 0$ and $\mathbf{P}(\text{tails}) > 0$.

Solution to Problem 1.34. The system may be viewed as a series connection of three subsystems, denoted 1, 2, and 3 in Fig. 1.19 in the text. The probability that the entire system is operational is $p_1 p_2 p_3$, where p_i is the probability that subsystem i is operational. Using the formulas for the probability of success of a series or a parallel system given in Example 1.24, we have

$$p_1 = p, \qquad p_3 = 1 - (1-p)^2,$$

So solution is :

$p[1-(1-p)^2][1-(1-p)(1-p(1-(1-p)^3))$

and

$$p_2 = 1 - (1-p)\big(1 - p\big(1 - (1-p)^3\big)\big).$$

Solution to Problem 1.35. Let A_i be the event that exactly i components are operational. The probability that the system is operational is the probability of the union $\cup_{i=k}^{n} A_i$, and since the A_i are disjoint, it is equal to

$$\sum_{i=k}^{n} \mathbf{P}(A_i) = \sum_{i=k}^{n} p(i),$$

where $p(i)$ are the binomial probabilities. Thus, the probability of an operational system is

$$\sum_{i=k}^{n} \binom{n}{i} p^i (1-p)^{n-i}.$$

Solution to Problem 1.49. A sum of 11 is obtained with the following 6 combinations:

$$(6,4,1)\ (6,3,2)\ (5,5,1)\ (5,4,2)\ (5,3,3)\ (4,4,3).$$

A sum of 12 is obtained with the following 6 combinations:

$$(6,5,1)\ (6,4,2)\ (6,3,3)\ (5,5,2)\ (5,4,3)\ (4,4,4).$$

Each combination of 3 distinct numbers corresponds to 6 permutations, while each combination of 3 numbers, two of which are equal, corresponds to 3 permutations. Counting the number of permutations in the 6 combinations corresponding to a sum of 11, we obtain $6+6+3+6+3+3 = 27$ permutations. Counting the number of permutations in the 6 combinations corresponding to a sum of 12, we obtain $6+6+3+3+6+1 = 25$ permutations. Since all permutations are equally likely, a sum of 11 is more likely than a sum of 12.

Note also that the sample space has $6^3 = 216$ elements, so we have $\mathbf{P}(11) = 27/216$, $\mathbf{P}(12) = 25/216$.

Solution to Problem 1.52. The probability that the 13th card is the first king to be dealt is the probability that out of the first 13 cards to be dealt, exactly one was a king, and that the king was dealt last. Now, given that exactly one king was dealt in the first 13 cards, the probability that the king was dealt last is just 1/13, since each "position" is equally likely. Thus, it remains to calculate the probability that there was exactly one king in the first 13 cards dealt. To calculate this probability we count the "favorable" outcomes and divide by the total number of possible outcomes. We first count the favorable outcomes, namely those with exactly one king in the first 13 cards dealt. We can choose a particular king in 4 ways, and we can choose the other 12 cards in $\binom{48}{12}$ ways, therefore there are $4 \cdot \binom{48}{12}$ favorable outcomes. There are $\binom{52}{13}$ total outcomes, so the desired probability is

$$\frac{1}{13} \cdot \frac{4 \cdot \binom{48}{12}}{\binom{52}{13}}. \quad \approx .03375\ldots$$

For an alternative solution, we argue as in Example 1.10. The probability that the first card is not a king is 48/52. Given that, the probability that the second is not a king is 47/51. We continue similarly until the 12th card. The probability that the 12th card is not a king, given that none of the preceding 11 was a king, is 37/41. (There are $52-11=41$ cards left, and $48-11=37$ of them are not kings.) Finally, the conditional probability that the 13th card is a king is 4/40. The desired probability is

$$\frac{48 \cdot 47 \cdots 37 \cdot 4}{52 \cdot 51 \cdots 41 \cdot 40}.$$

Solution to Problem 1.53. Suppose we label the classes A, B, and C. The probability that Joe and Jane will both be in class A is the number of possible combinations for class A that involve both Joe and Jane, divided by the total number of combinations for class A. Therefore, this probability is

$$\frac{\binom{88}{28}}{\binom{90}{30}}. \quad \approx .3258\ldots$$

Since there are three classes, the probability that Joe and Jane end up in the same class is

$$3 \cdot \frac{\binom{88}{28}}{\binom{90}{30}}.$$

A much simpler solution is as follows. We place Joe in one class. Regarding Jane, there are 89 possible "slots", and only 29 of them place her in the same class as Joe. Thus, the answer is 29/89, which turns out to agree with the answer obtained earlier.

Solution to Problem 1.54. (a) Since the cars are all distinct, there are 20! ways to line them up.

(b) To find the probability that the cars will be parked so that they alternate, we count the number of "favorable" outcomes, and divide by the total number of possible outcomes found in part (a). We count in the following manner. We first arrange the US cars in an ordered sequence (permutation). We can do this in 10! ways, since there are 10 distinct cars. Similarly, arrange the foreign cars in an ordered sequence, which can also be done in 10! ways. Finally, interleave the two sequences. This can be done in two different ways, since we can let the first car be either US-made or foreign. Thus, we have a total of $2 \cdot 10! \cdot 10!$ possibilities, and the desired probability is

$$\frac{2 \cdot 10! \cdot 10!}{20!}. \quad \approx .0000108\ldots$$

Note that we could have solved the second part of the problem by neglecting the fact that the cars are distinct. Suppose the foreign cars are indistinguishable, and also that the US cars are indistinguishable. Out of the 20 available spaces, we need to choose 10 spaces in which to place the US cars, and thus there are $\binom{20}{10}$ possible outcomes. Out of these outcomes, there are only two in which the cars alternate, depending on whether we start with a US or a foreign car. Thus, the desired probability is $2/\binom{20}{10}$, which coincides with our earlier answer.

Solution to Problem 1.56. (a) There are $\binom{8}{4}$ ways to pick 4 lower level classes, and $\binom{10}{3}$ ways to choose 3 higher level classes, so there are

$$\binom{8}{4}\binom{10}{3} = 8400$$

valid curricula.

(b) This part is more involved. We need to consider several different cases:

(i) Suppose we do not choose L_1. Then both L_2 and L_3 must be chosen; otherwise no higher level courses would be allowed. Thus, we need to choose 2 more lower level classes out of the remaining 5, and 3 higher level classes from the available 5. We then obtain $\binom{5}{2}\binom{5}{3}$ valid curricula. $= 100$

(ii) If we choose L_1 but choose neither L_2 nor L_3, we have $\binom{5}{3}\binom{5}{3}$ choices. $= 100$

(iii) If we choose L_1 and choose one of L_2 or L_3, we have $2 \cdot \binom{5}{2}\binom{5}{3}$ choices. This is $= 200$ because there are two ways of choosing between L_2 and L_3, $\binom{5}{2}$ ways of choosing 2 lower level classes from $L_4, \ldots, L_8$, and $\binom{5}{3}$ ways of choosing 3 higher level classes from $H_1, \ldots, H_5$.

(iv) Finally, if we choose L_1, L_2, and L_3, we have $\binom{5}{1}\binom{10}{3}$ choices. $= 600$

Note that we are not double counting, because there is no overlap in the cases we are considering, and furthermore we have considered every possible choice. The total is obtained by adding the counts for the above four cases. $\Big\}$ 1000

Solution to Problem 1.57. Let us fix the order in which letters appear in the sentence. There are 26! choices, corresponding to the possible permutations of the 26-letter alphabet. Having fixed the order of the letters, we need to separate them into words. To obtain 6 words, we need to place 5 separators ("blanks") between the letters. With 26 letters, there are 25 possible positions for these blanks, and the number of choices is $\binom{25}{5}$. Thus, the desired number of sentences is $\cancel{25!\binom{25}{5}}$. Generalizing, the number of sentences consisting of w nonempty words using exactly once each letter

$26!\binom{25}{5} \approx 2\times 10^{31}$

from a l-letter alphabet is equal to

$$l!\binom{l-1}{w-1}.$$

Solution to Problem 1.58. (a) The sample space consists of all ways of drawing 7 elements out of a 52-element set, so it contains $\binom{52}{7}$ possible outcomes. Let us count those outcomes that involve exactly 3 aces. We are free to select any 3 out of the 4 aces, and any 4 out of the 48 remaining cards, for a total of $\binom{4}{3}\binom{48}{4}$ choices. Thus,

$$\mathbf{P}(7 \text{ cards include exactly 3 aces}) = \frac{\binom{4}{3}\binom{48}{4}}{\binom{52}{7}}. \quad \approx .0058$$

(b) Proceeding similar to part (a), we obtain

$$\mathbf{P}(7 \text{ cards include exactly 2 kings}) = \frac{\binom{4}{2}\binom{48}{5}}{\binom{52}{7}}. \quad \approx .07679$$

(c) If A and B stand for the events in parts (a) and (b), respectively, we are looking for $\mathbf{P}(A \cup B) = \mathbf{P}(A) + \mathbf{P}(B) - \mathbf{P}(A \cap B)$. The event $A \cap B$ (having exactly 3 aces and exactly 2 kings) can occur by choosing 3 out of the 4 available aces, 2 out of the 4 available kings, and 2 more cards out of the remaining 44. Thus, this event consists of $\binom{4}{3}\binom{4}{2}\binom{44}{2}$ distinct outcomes. Hence,

$$\mathbf{P}(7 \text{ cards include 3 aces and/or 2 kings}) = \frac{\binom{4}{3}\binom{48}{4} + \binom{4}{2}\binom{48}{5} - \binom{4}{3}\binom{4}{2}\binom{44}{2}}{\binom{52}{7}}. \quad \approx .0827$$

~~and the desired probability is~~

$$\cancel{\frac{\binom{k}{n}\binom{100-k}{m-n}}{\binom{100}{m}}}.$$

Solution to Problem 1.60. The size of the sample space is the number of different ways that 52 objects can be divided in 4 groups of 13, and is given by the multinomial formula

$$\frac{52!}{13!\,13!\,13!\,13!}.$$

There are 4! different ways of distributing the 4 aces to the 4 players, and there are

$$\frac{48!}{12!\,12!\,12!\,12!}$$

different ways of dividing the remaining 48 cards into 4 groups of 12. Thus, the desired probability is

$$\frac{4!\dfrac{48!}{12!\,12!\,12!\,12!}}{\dfrac{52!}{13!\,13!\,13!\,13!}}. \quad \approx .10549$$

An alternative solution can be obtained by considering a different, but probabilistically equivalent method of dealing the cards. Each player has 13 slots, each one of which is to receive one card. Instead of shuffling the deck, we place the 4 aces at the top, and start dealing the cards one at a time, with each free slot being equally likely to receive the next card. For the event of interest to occur, the first ace can go anywhere; the second can go to any one of the 39 slots (out of the 51 available) that correspond to players that do not yet have an ace; the third can go to any one of the 26 slots (out of the 50 available) that correspond to the two players that do not yet have an ace; and finally, the fourth, can go to any one of the 13 slots (out of the 49 available) that correspond to the only player who does not yet have an ace. Thus, the desired probability is

$$\frac{39 \cdot 26 \cdot 13}{51 \cdot 50 \cdot 49}.$$

By simplifying our previous answer, it can be checked that it is the same as the one obtained here, thus corroborating the intuitive fact that the two different ways of dealing the cards are probabilistically equivalent.