Introduction to Probability, Fall 2013

Math 30530 Section 01

Homework 5 — solutions

1. Chapter 2, problem 16 — see supplementary solutions file

2. Chapter 2, problem 17 — see supplementary solutions file

3. Chapter 2, problem 18 — see supplementary solutions file

4. Chapter 2, problem 20 — Let X be the number of bars you need to buy to win. Then X is a geometric random variable with parameter p. It takes values $1, 2, \ldots$, and $\Pr(X = k) = (1-p)^{k-1}p$.

   The expected value of X is

$$
\begin{aligned}
E(X) &= 1p + 2(1-p)p + 3(1-p)^2p + 4(1-p)^3p + \ldots \\
&= p\left(1 + 2(1-p) + 3(1-p)^2 + 4(1-p)^3 + \ldots\right) \\
&= p\frac{1}{(1-(1-p))^2} \\
&= \frac{1}{p},
\end{aligned}
$$

   where the second-from-last equality uses the first differentiation trick described below with $x = 1-p$.

   The expected value of X^2 is

$$
\begin{aligned}
E(X^2) &= 1^2p + 2^2(1-p)p + 3^2(1-p)^2p + 4^2(1-p)^3p + \ldots \\
&= p\left(1^2 + 2^2(1-p) + 3^2(1-p)^2 + 4^2(1-p)^3 + \ldots\right) \\
&= p\left([1.0+1] + [2.1+2](1-p) + [3.2+3](1-p)^2 + [4.3+4](1-p)^3 + \ldots\right) \\
&= p\left(\left(2.1(1-p) + 3.2(1-p)^2 + 4.3(1-p)^3 + \ldots\right) + \left(1 + 2(1-p) + 3(1-p)^2 + 4(1-p)^3 + \right.\right. \\
&= p(1-p)\left(2.1 + 3.2(1-p) + 4.3(1-p)^2 + \ldots\right) + p\left(1 + 2(1-p) + 3(1-p)^2 + 4(1-p)^3 + \right. \\
&= p(1-p)\frac{2}{(1-(1-p))^3} + p\frac{1}{(1-(1-p))^2} \\
&= \frac{2-p}{p^2}
\end{aligned}
$$

   where the second-from-last equality uses both differentiation tricks described below, with $x = 1-p$.

So

$$\mathrm{Var}(X) = E(X^2) - E(X)^2 = \frac{2-p}{p^2} - \frac{1}{p^2} = \frac{1-p}{p^2}.$$

(Differentiation tricks: since $1 + x + x^2 + \ldots + x^k + \ldots = 1/(1-x)$, by differentiating both sides we get $1+2x+3x^2+4x^3+\ldots+kx^{k-1}+\ldots = 1/(1-x)^2$, and by differentiating again we get $2 + 6x + 12x^2 + 20x^3 + \ldots + k(k-1)x^{k-2} + \ldots = 2/(1-x)^3$)

5. Chapter 2, problem 22 — see supplementary solutions file. For part a) we just observe that the experiment we are doing leads to a geometric random variable with success probability $p(1-q)+(1-p)q$, and use formulae for expectation and variance calculated in the problem Chapter 2.20. For part b), we want to know the probability that the experiment ends with first coin heads, second coin tails. Knowing that the experiment must end EITHER with first coin heads, second coin tails OR with first coin tails, second coin heads, the "memoryless" property of the geometric means that the probability we are trying to calculate is just the probability that, when I flip the two coins, I see first coin heads, second coin tails, given the conditional information that I see EITHER first coin heads, second coin tails OR first coin tails, second coin heads.

6. Chapter 2, problem 23 — First, part a):

   The possible values of X, the number of tosses, are $2, 3, 4, \ldots$. The two outcomes that lead to $X = k$ are $HTHTHT\ldots HTHH$ and $THTHTH\ldots THTT$, so $\Pr(X = k) = 2(1/2)^k = (1/2)^{k-1}$. So the mass function of X is: $p_X(x) = (1/2)^{x-1}$ if $x = 2, 3, 4\ldots$, and $p_X(x) = 0$ otherwise. So

   $$E(X) = 2(1/2) + 3(1/4) + 4(1/8) + \ldots.$$

   This looks very like the sum for the expectation of a geometric with probability $1/2$, which is

   $$(1/2) + 2(1/4) + 3(1/8) + \ldots = 2.$$

   Indeed, multiplying the above sum by 2 gives

   $$1 + 2(1/2) + 3(1/4) + 4(1/8) + \ldots = 4,$$

   so

   $$E(X) = 2(1/2) + 3(1/4) + 4(1/8) + \ldots = 4 - 1 = 3.$$

   For the variance, look at $E(X^2)$:

   $$E(X^2) = 2^2(1/2) + 3^2(1/4) + 4^2(1/8) + \ldots.$$

   Looking back at the problem of computing the variance of a geometric, we see that this is very close to our calculation of the expectation of the square of the geometric, when $p = 1/2$:

   $$1^2(1/2) + 2^2(1/4) + 3^2(1/8) + 4^2(1/16) + \ldots = 6.$$

   Indeed, multiplying the above sum by 2 gives

   $$1^2 + 2^2(1/2) + 3^2(1/4) + 4^2(1/8) + \ldots = 12,$$

so

$$E(X^2) = 2^2(1/2) + 3^2(1/4) + 4^2(1/8) + \ldots = 12 - 1 = 11,$$

and $\mathrm{Var}(X) = 11 - (3)^2 = 2$. **SUMMARY**: Expectation is 3, variance is 2.

Now, part b):

Let X be number of tosses. Possible values for X are $2, 3, 4, \ldots$. How many ways to succeed in k tosses? We must end HT, and have no earlier HT, but we can have an earlier TH (but only one ... to have a second we would need a HT in between). So we start with some string (maybe empty) of tails, flip once to heads, then end with a flip to tails. There are $k-1$ such strings (for example, when $k = 6$ we have five possibilities, HHHHHT, THHHHT, TTHHHT, TTTHHT and TTTTHT). So the mass function of X is: $p_X(x) = (x-1)(1/2)^x$ if $x = 2, 3, 4 \ldots$, and $p_X(x) = 0$ otherwise. So

$$E(X) = 2.1(1/4) + 3.2(1/8) + 4.3(1/16) + \ldots.$$

We sum this using the differentiation trick:

$$\frac{2}{(1-x)^3} = 2.1 + 3.2x + 4.3x^2 + \ldots.$$

Applying with $x = 1/2$ we get

$$16 = 2.1 + 3.2(1/2) + 4.3(1/2)^2 + \ldots.$$

Dividing both sides by 4,

$$E(X) = 2.1(1/4) + 3.2(1/8) + 4.3(1/16) + \ldots = 4.$$

SUMMARY: Expectation is 4.

7. Use the Taylor series of e^x to figure out the expected value of a Poisson random variable with parameter λ. — Remember $e^x = 1 + x + x^2/2 + x^3/3! + \ldots$. So for $X \sim \mathrm{Poisson}(\lambda)$ (with X taking values 0, 1, 2, …, value k with probability $\lambda^k/k! \times e^{-\lambda}$)

$$\begin{aligned} E(X) &= \sum_{k=0}^{\infty} k \frac{\lambda^k}{k!} e^{-\lambda} \\ &= \lambda e^{-\lambda} \sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!} \\ &= \lambda e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \\ &= \lambda e^{-\lambda} e^{\lambda} \\ &= \lambda, \end{aligned}$$

as required.

8. Let X be a negative Binomial random variable that counts the number of independent repetitions of a trial needed to obtain a total of m successes, when each individual trial has success probability p. Show that

$$\Pr(X > k) = \sum_{j=0}^{m-1} \binom{k}{j} p^j (1-p)^{k-j}.$$

(When $m = 1$ X becomes geometric with parameter p, and the equation reduces to $\Pr(X > k) = (1-p)^k$, which we discussed in class.) — Event "X a negative binomial with parameters m and p, $X > k$" is the same as "after k independent repetition of the trial, still have not reached m success", which is the same as "after k independent repetition of the trial, the number of success is m-1 or fewer", which is the same as "Y a binomial random variable with parameters k and p, $Y \leq m-1$". The left-hand side is the probability of the event in its first description, and the right-hand side is the probability of the event in its second description. Since the two events are the same, so are the two sides of the equation.

9. In the Banach matchbox problem, suppose that we stop not when we first reach into a pocket and find an empty matchbox, but instead when we first reach into a pocket and realize that the match we are taking out is the last match in that pocket. Find the probability that at this moment the other pocket has exactly k matches. (Be sure to say what the possible values of k are). — Let's say we have N matches in each pocket. The possible numbers of matches in the other pocket, when the reached-into pocket has its last match removed (and is the first pocket for which this happens), are $k = N, N-1, \ldots, 1$ (NOT 0 this time). For each such k, the outcomes that lead to that number of matches in the other pocket are encoded by EITHER strings of length $2N-k$, ending R, in the first $2N-k-1$ places exactly $N-1$ R's; OR strings of length $2N-k$, ending L, in the first $2N-k-1$ places exactly $N-1$ L's. There are

$$2\binom{2N-k-1}{N-1}$$

such strings, each with probability $(1/2)^{2N-k}$. So

$$\Pr(X = k) = \frac{\binom{2N-k-1}{N-1}}{2^{2N-k-1}}.$$

10. Here is a winning strategy for Roulette: first, bet \$1 on red (so you win your dollar back, plus one more dollar, with probability 18/38, and you lose your dollar with probability 20/38). If red does come up, take your winnings and run. If not, then make additional bets on red for each of the next two spins of the wheel, and quit after that. Let X denote your net winnings at the end of this process.

   (a) Calculate $\Pr(X > 0)$ (your answer should be some number $> .5$, which shows that this is a winning strategy — you come out on top more times than you come out behind, on average) — We encode the possible outcomes of the strategy by strings of W (meaning you won on that particular spin) and L (meaning you lost). Here are the 5 possibilities, together with the probability of each (calculated by independence of each roll), together with the net winning (+1 on each spin that you won, −1 on each that you lost):

W, $p = 18/38 \approx .474$, +1 LWW, $p = (20/38)(18/38)(18/38) \approx .118$, +1 LWL, $p = (20/38)(18/38)(20/38) \approx .131$, −1 LLW, $p = (20/38)(20/38)(18/38) \approx .131$, −1 LLL, $p = (20/38)(20/38)(20/38) \approx .146$, −3

So $\Pr(X > 0) \approx .474 + .118 = .592$.

(b) Calculate the expected proportion of time in which you come out on top (with positive net winnings) while employing this strategy. — around 59.2% of the time (proportion .592)

(c) Calculate $E(X)$, the expected net winnings while employing this strategy. —

$$E(X) = 1(.592) - 1(.262) - 3(.146) = -.108$$

(so you lose money on average with my "winning" strategy :()

11. Four busses carrying 148 students arrive at a school. The busses carry 40, 33, 25 and 50 students. A student is selected at random from among the 148. Let X be the number of students on the bus of the randomly selected student. A random driver (from among the four drivers) is selected at random. Let Y be the number of students on the bus of the randomly selected driver.

(a) Without doing the calculations, which of $E(X)$, $E(Y)$ do you think might be bigger, and why? — I expect $E(X)$ to be bigger, because by picking a student at random, I'm more likely to pick a student from a bus that has a large number of students on it (those busses give me more students to choose from).

(b) Check your intuition by calculating $E(X)$ and $E(Y)$ and determining which is actually bigger. —

$$E(X) = 40\left(\frac{40}{148}\right) + 33\left(\frac{33}{148}\right) + 25\left(\frac{25}{148}\right) + 50\left(\frac{50}{148}\right)$$

which is ≈ 39.28. On the other hand

$$E(Y) = 40\left(\frac{1}{4}\right) + 33\left(\frac{1}{4}\right) + 25\left(\frac{1}{4}\right) + 50\left(\frac{1}{4}\right)$$

which is exactly 37. So indeed $E(X) > E(Y)$.

12. On Monday afternoon you have \$1000 in your brokerage account, and nickel is selling for \$2 an ounce. You assess that on Tuesday morning, nickel will be worth either \$1 an ounce, or \$4 an ounce, each option equally likely. The only think you can do with your brokerage account is buy some nickel (not necessarily \$1000 worth) or keep money in the account.

(a) Suppose your objective is to maximize the expected value of your portfolio (brokerage account balance plus value of nickel holdings) on late Tuesday morning. What should your Monday afternoon/Tuesday morning strategy be? — I don't change my net worth by buying/selling on Tuesday morning, so my strategy should be to use x dollars on Monday afternoon to buy $x/2$ ounces of nickel. My net worth on Tuesday is then $1000 - x + (x/2)1 = 1000 - x/2$ with probability .5 (if nickel drops in value), and $1000 - x + (x/2)4 = 1000 + x$ with probability .5 (if nickel rises in value). So my expected net worth is

$$\frac{1}{2}\left(1000 - \frac{x}{2}\right) + \frac{1}{2}(1000 + x) = 1000 + \frac{x}{4}.$$

I should choose x as large as possible to maximize this, that is, I should spend ALL my money on MONDAY to buy nickel.

(b) Suppose your objective is to maximize the expected amount (not value) of nickel you hold on late Tuesday morning. What should your Monday afternoon/Tuesday morning strategy be? — In this case there is no point leaving any money in the account (I could just use it to buy nickel and increase my nickel holdings). So my strategy should be to use x dollars to buy nickel on Monday, and the remaining $1000 - x$ on Tuesday. With this strategy, I end up with $(x/2) + (1000 - x) = 1000 - (x/2)$ ounces of nickel with probability 1/2, and with $(x/2) + (1000 - x)/4 = 250 + (x/4)$ with probability 1/2, so my expected nickel holding is

$$\frac{1}{2}\left(1000 - \frac{x}{2}\right) + \frac{1}{2}\left(250 + \frac{x}{4}\right) = 625 - \frac{x}{8}.$$

I should choose x as small as possible to maximize this, that is, I should spend ALL my money on TUESDAY to buy nickel.

13. Suppose that X is a random variable that only takes on values $0, 1, 2, \ldots$. Show that

$$E(X) = \sum_{i=1}^{\infty} \Pr(X \geq i).$$

— We know

$$E(X) = \sum_{i=0}^{\infty} i \Pr(X = i).$$

For each k, the probability $\Pr(X = k)$ appears exactly k times in this sum. So we will show that the claimed formula is correct by showing that when we replace each $\Pr(X \geq i)$ in the claimed formula with $\Pr(X = i) + \Pr(X = i+1) + \Pr(X = i+2) + \ldots$, then for each k, $\Pr(X = k)$ appears exactly k times in the formula. For each k, it appears in the expansion of $\Pr(X \geq 1)$, and $\Pr(X \geq 2)$, and $\Pr(X \geq 3)$, etc., all the way up to $\Pr(X \geq k)$, and in no more expanded-out $\Pr(X \geq i)$'s. So indeed it appears exactly k times, and the two formulas are the same.

We could also write, quite compactly,

$$\sum_{i=1}^{\infty} \Pr(X \geq i) = \sum_{i=1}^{\infty} \sum_{j=i}^{\infty} \Pr(X = j) = \sum_{j=1}^{\infty} \sum_{i=1}^{j} \Pr(X = j) = \sum_{j=1}^{\infty} j \Pr(X = j) = E(X)$$

14. Let X be a binomial random variable with parameters n and p. Show that

$$E(1/(X+1)) = \frac{1-(1-p)^{n+1}}{(n+1)p}.$$

— The mass function of the binonial is $\Pr(X = k) = \binom{n}{k}p^k(1-p)^{n-k}$ for $k = 0, 1, 2, \ldots, n$, and 0 other wise. Since we already know the answer we are aiming for, we can try to prove it by showing that $(n+1)pE(1/(X+1)) + (1-p)^{n+1} = 1$. In what follows I'll use $\frac{n+1}{k+1}\binom{n}{k} = \binom{n+1}{k+1}$, which is the same as $(n+1)\binom{n}{k} = (k+1)\binom{n+1}{k+1}$, which is the committee-chair result with $n+1$ people, and a committee of size $k+1$. We have

$$\begin{aligned}
(n+1)pE(1/(X+1)) + (1-p)^{n+1} &= (n+1)p\left(\sum_{k=0}^{n}\frac{1}{k+1}\binom{n}{k}p^k(1-p)^{n-k}\right) + (1-p)^{n+1} \\
&= \left(\sum_{k=0}^{n}\frac{n+1}{k+1}\binom{n}{k}p^{k+1}(1-p)^{n-k}\right) + (1-p)^{n+1} \\
&= \left(\sum_{k=0}^{n}\binom{n+1}{k+1}p^{k+1}(1-p)^{(n+1)-(k+1)}\right) + (1-p)^{n+1} \\
&= \left(\sum_{k=1}^{n+1}\binom{n+1}{k}p^{k}(1-p)^{(n+1)-k}\right) + (1-p)^{n+1} \\
&= \sum_{k=0}^{n+1}\binom{n+1}{k}p^{k}(1-p)^{(n+1)-k} \\
&= 1
\end{aligned}$$

as required, the last equality using the binomial theorem.