Math 30530, Fall 2013, Homework 5, supplementary solutions file

2.16

Solution to Problem 2.16. (a) The scalar a must satisfy

$$1 = \sum_x p_X(x) = \frac{1}{a} \sum_{x=-3}^{3} x^2,$$

so

$$a = \sum_{x=-3}^{3} x^2 = (-3)^2 + (-2)^2 + (-1)^2 + 1^2 + 2^2 + 3^2 = 28.$$

We also have $\mathbf{E}[X] = 0$ because the PMF is symmetric around 0.

(b) If $z \in \{1, 4, 9\}$, then

$$p_Z(z) = p_X(\sqrt{z}) + p_X(-\sqrt{z}) = \frac{z}{28} + \frac{z}{28} = \frac{z}{14}.$$

Otherwise $p_Z(z) = 0$.

(c) $\text{var}(X) = \mathbf{E}[Z] = \sum_z z p_Z(z) = \sum_{z \in \{1,4,9\}} \frac{z^2}{14} = 7.$

(d) We have

$$\begin{aligned}
\text{var}(X) &= \sum_x (x - \mathbf{E}[X])^2 p_X(x) \\
&= 1^2 \cdot \big(p_X(-1) + p_X(1)\big) + 2^2 \cdot \big(p_X(-2) + p_X(2)\big) + 3^2 \cdot \big(p_X(-3) + p_X(3)\big) \\
&= 2 \cdot \frac{1}{28} + 8 \cdot \frac{4}{28} + 18 \cdot \frac{9}{28} \\
&= 7.
\end{aligned}$$

2.17

Solution to Problem 2.17. If X is the temperature in Celsius, the temperature in Fahrenheit is $Y = 32 + 9X/5$. Therefore,

$$\mathbf{E}[Y] = 32 + 9\mathbf{E}[X]/5 = 32 + 18 = 50.$$

Also

$$\text{var}(Y) = (9/5)^2\text{var}(X),$$

where $\text{var}(X)$, the square of the given standard deviation of X, is equal to 100. Thus, the standard deviation of Y is $(9/5) \cdot 10 = 18$. Hence a normal day in Fahrenheit is one for which the temperature is in the range $[32, 68]$.

2.18

Solution to Problem 2.18. We have

$$p_X(x) = \begin{cases} 1/(b-a+1), & \text{if } x = 2^k, \text{ where } a \le k \le b,\ k \text{ integer}, \\ 0, & \text{otherwise}, \end{cases}$$

and

$$\mathbf{E}[X] = \sum_{k=a}^{b} \frac{1}{b-a+1} 2^k = \frac{2^a}{b-a+1}(1 + 2 + \cdots + 2^{b-a}) = \frac{2^{b+1} - 2^a}{b-a+1}.$$

Similarly,

$$\mathbf{E}[X^2] = \sum_{k=a}^{b} \frac{1}{b-a+1}(2^k)^2 = \frac{4^{b+1} - 4^a}{3(b-a+1)},$$

and finally

$$\text{var}(X) = \frac{4^{b+1} - 4^a}{3(b-a+1)} - \left(\frac{2^{b+1} - 2^a}{b-a+1}\right)^2.$$

2.22

Solution to Problem 2.22. (a) Let X be the number of tosses until the game is over. Noting that X is geometric with probability of success

$$\mathbf{P}\big(\{HT, TH\}\big) = p(1-q) + q(1-p),$$

we obtain

$$p_X(k) = \big(1 - p(1-q) - q(1-p)\big)^{k-1}\big(p(1-q) + q(1-p)\big), \qquad k = 1, 2, \ldots$$

Therefore

$$\mathbf{E}[X] = \frac{1}{p(1-q) + q(1-p)}$$

and

$$\text{var}(X) = \frac{pq + (1-p)(1-q)}{\big(p(1-q) + q(1-p)\big)^2}.$$

(b) The probability that the last toss of the first coin is a head is

$$\mathbf{P}\big(HT \mid \{HT, TH\}\big) = \frac{p(1-q)}{p(1-q) + (1-q)p}.$$