Introduction to Probability, Fall 2013

Math 30530 Section 01

Homework 8 — Solutions

1. Verify that the variance of the standard normal random variable Z with parameters $\mu = 0$ and $\sigma^2 = 1$ is indeed 1, by explicitly computing $\int_{-\infty}^{\infty}(1/\sqrt{2\pi})x^2e^{-x^2/2}dx$. You can assume that $E(Z) = 0$ and that the density function for Z, $f(x) = (1/\sqrt{2\pi})e^{-x^2/2}$, is indeed a density function, that is that $\int_{-\infty}^{\infty}(1/\sqrt{2\pi})e^{-x^2/2}dx = 1$.

   Solution: We know $E(Z) = 0$, so

   $$\text{Var}(Z) = E(Z^2) - (E(Z))^2 = E(Z^2) = \int_{-\infty}^{\infty}(1/\sqrt{2\pi})x^2e^{-x^2/2}dx.$$

   Use integration by parts with $u = x$ and $dv = (1/\sqrt{2\pi})xe^{-x^2/2}dx$, so $du = dx$ and $v = -(1/\sqrt{2\pi})e^{-x^2/2}$. We get

   $$\begin{aligned}\int_{-\infty}^{\infty}(1/\sqrt{2\pi})x^2e^{-x^2/2}dx &= \left[-(1/\sqrt{2\pi})xe^{-x^2/2}\right]_{x=-\infty}^{\infty} - \int_{-\infty}^{\infty} -(1/\sqrt{2\pi})e^{-x^2/2}dx \\ &= [0-0] + \int_{-\infty}^{\infty}(1/\sqrt{2\pi})e^{-x^2/2}dx \\ &= 1,\end{aligned}$$

   the last line using $\int_{-\infty}^{\infty}(1/\sqrt{2\pi})e^{-x^2/2}dx = 1$.

2. Chapter 3, problems 11, 12, 13 — see supplemental solutions file.

3. The temperature of a steel rod four hours after tempering is known to be normally distributed with mean 75 degree Celsius and standard deviation 25.

   (a) Compute the probability that a rod has temperature above 105 degrees Celsius four hours after tempering.

   Solution: Let X be temp. of rod after 4 hours. $X \sim \text{Normal}(75, 25^2)$. Want $\Pr(X > 105) = \Pr(Z > 1.2) = .1151$.

   (b) I can begin using a rod after tempering once its temperature has dropped below 105 degrees Celsius. If 10 rods are set aside for four hours after tempering, whats the probability that I can use at least 7 of them at this time?

   Solution: Let Y be number I can use. By previous part, $Y \sim \text{Binomial}(10, .8849)$. Want $\Pr(Y \geq 7) = .9792$ (from a binomial calculator).

4. A soft drink machine can be regulated so that it discharges an average of μ ounces per cup. If the amount the machine dispenses is modeled by a normal random variable with standard deviation 0.3 ounces, find a value of μ such that 8-ounce cups will over flow only 1% of the time.

   Solution: Let X be amount dispensed; $X \sim \text{Normal}(\mu, .3)$. Want $\Pr(X \geq 8) = .01$. From table, $\Pr(Z > 2.33) = .01$ (Z a standard normal). So, we want to choose μ so that 8 is 2.33 standard deviations above μ; i.e, $\mu = 8 - (2.33)(.3) = .7301$.

5. The distribution of resistance for resistors of a certain type is known to be normal. 9.85% of all resistors have a resistance exceeding 10.257 Ohms, and 5.05% have resistance smaller than 9.671 Ohms. What are the mean value and standard deviation of the resistance distribution?

   Solution: Let X be distribution of resistors; $X \sim \text{Normal}(\mu, \sigma)$. Know:

$$\Pr(X > 10.257) = .0985, \quad \text{so} \quad \Pr(Z > (10.257 - \mu)/\sigma) = .0985.$$

   From normal table, $\Pr(Z > 1.29) = .0985$, so $(10.257 - \mu)/\sigma = 1.29$.

   Also know:

$$\Pr(X > 9.671) = .0505, \quad \text{so} \quad \Pr(Z > (9.671 - \mu)/\sigma) = .0505.$$

   From normal table, $\Pr(Z < -1.64) = .0505$, so $(9.671 - \mu)/\sigma = -1.64$.

   Solving these two equations get $\mu = 10$ and $\sigma = .2$.

6. Chapter 3, problem 15 — see supplemental solutions file.

7. My dog Casey has run off into the forest. Painful past experience has taught me that the time until Casey is sprayed by a skunk is exponentially distributed with an average (expected value) of 2 hours. If the time it takes me to run back home and return with Casey's favorite squeak-toy ranges between 20 and 40 minutes, and is uniformly distributed over that interval, Calculate the probability that I succeed in luring Casey back with her favorite toy before she gets sprayed by a skunk. (Assume that Casey comes back to me the moment I return with the squeak-toy).

   Solution: X = time 'til skunk sprays (exponential with $\lambda = 1/2$ if units are hours); Y = time until I return (uniform with $a = 1/3$, $b = 2/3$ if units are hours). Joint density of X and Y is product of individual densities (since X, Y independent), so it's 0 except on the strip $1/3 \leq y \leq 2/3$, $0 \leq x \leq \infty$; on this strip the joint density is

$$f(x, y) = \left(\frac{1}{2}e^{-x/2}\right)\left(\frac{1}{\frac{2}{3} - \frac{1}{3}}\right) = \frac{3}{2}e^{-x/2}.$$

   We want $\Pr(X > Y)$; the part of the strip with $x > y$ is a razor-shaped area described by $1/3 \leq y \leq 2/3$, and for each such y, $y \leq x \leq \infty$. So

$$\Pr(X > Y) = \int_{y=1/3}^{2/3} \int_{x=y}^{\infty} \frac{3}{2}e^{-x/2}\, dxdy \approx .7797.$$

8. An introverted professor X rarely turns her face away from the blackboard. The moment when she first faces her students is equally likely to occur at any point during her hour-long lecture. X's student, Y, is very busy. He's always at least 10 minutes late, though he always manages to get to class (at a completely random moment) before the lecture is halfway through. How likely is it that when X faces the class for the first time, she'll see Y eagerly taking notes?

   Solution: X = time 'til Prof. turns around (uniform with $a = 0$, $b = 60$ if units are minutes); Y = time until student shows up (uniform with $a = 10$, $b = 30$ if units are minutes). Joint density of X and Y is product of individual densities (since X, Y independent), so it's 0 except on the strip $0 \leq x \leq 60$, $10 \leq y \leq 30$; on this strip the joint density is
$$f(x, y) = \left(\frac{1}{60-0}\right)\left(\frac{1}{30-10}\right) = \frac{1}{1200}.$$
   We want $\Pr(X > Y)$; the part of the strip with $x > y$ is a razor-shaped area described by $10 \leq y \leq 30$, and for each such y, $y \leq x \leq 60$. So
$$\Pr(X > Y) = \int_{y=10}^{30} \int_{x=y}^{60} \frac{1}{1200}\, dxdy = 2/3.$$
   (Or: we could have just computed the area of the razor, which is 800, and divided it by the total area 1200, since the joint distribution on the strip is uniform).

9. The joint density of a pair of random variables X, Y is
$$f(x, y) = \begin{cases} Cxe^{-4y} & \text{if } 0 \leq x \leq 4 \text{ and } 0 \leq y \leq \infty \\ 0 & \text{otherwise.} \end{cases}$$

   (a) Find C.

   Solution: We calculate
$$\int_{x=0}^{4} \int_{y=0}^{\infty} xe^{-4y}\, dydx = 2,$$
   so $C = 1/2$.

   (b) Are X and Y independent? Explain.

   Solution: It seems so, since the joint density is defined on a rectangle, and factors into a part involving x and a part involving y. We confirm by calculating
$$\int_{y=0}^{\infty} \frac{xe^{-4y}}{2}\, dy = \frac{x}{8}$$
   and
$$\int_{x=0}^{4} \frac{xe^{-4y}}{2}\, dx = 4e^{-4y}.$$

This shows that the marginal density of X is $\frac{x}{8}$ if $0 \leq x \leq 4$ (and is 0 otherwise), and that the marginal density of Y is $4e^{-4y}$ if $0 \leq y \leq \infty$ (and is 0 otherwise). The product of these two marginals is exactly the given joint density, so by definition of independence, the two random variables are independent.

(c) Calculate $\Pr(X + Y \geq 4)$.

Solution: We need to do the double integral of the joint density over that part of the region $0 \leq x \leq 4$, $0 \leq y \leq \infty$, where $x + y \geq 4$. This turns out to be:

$$\int_{x=0}^{4} \int_{y=4-x}^{infty} \frac{xe^{-4y}}{2} \, dydx \approx .117.$$

(d) Find the marginal density of X.

Solution: We've already done this earlier:

$$f_X(x) = \int_{y=-\infty}^{\infty} f(x,y) \; dy = \begin{cases} 0 & \text{if } x < 0 \text{ or } x > 4, \\ \int_{y=0}^{\infty} \frac{xe^{-4y}}{2} \; dy = \frac{x}{8} & \text{if } 0 \leq x \leq 4. \end{cases}$$

10. Let X and Y be independent uniform random variables, both taking values between -1 and 1. Find the probability that the quadratic equation

$$t^2 - Xt + Y = 0$$

has real roots.

Solution: The condition for real roots is $X^2 \geq 4Y$ of $Y \leq X^2/4$. So, the probability of real roots is the double integral, over that part of the square $[-1,1] \times [-1,1]$ where $y \leq x^2/4$, of the joint density of X and Y, which is the constant function $1/4$ on the square.

A picture shows that this probability is calculated as

$$\int_{x=-1}^{1} \int_{y=-1}^{x^2/4} \frac{1}{4} \; dydx \approx .5417.$$