# Introduction to Probability, Fall 2013

## Math 30530 Section 01

## Homework 9 — solutions

1. (a) Let $X$ be an exponential random variable with parameter $\lambda_1$, and $Y$ be an exponential random variable with parameter $\lambda_2$. If $X$ and $Y$ are independent, compute the density function of $Z = \min\{X, Y\}$, and show that it is exactly the same as the density function of the exponential random variable with parameter $\lambda_1 + \lambda_2$.

   **Solution**: Range of values for $Z$: 0 to $\infty$. For $0 \leq z \leq \infty$,

$$\begin{aligned}
\Pr(Z \leq z) &= 1 - \Pr(Z \geq z) \\
&= 1 - \Pr(\min\{X, Y\} \geq z) \\
&= 1 - \Pr(X \geq z \text{ and } Y \geq z) \\
&= 1 - \Pr(X \geq z)(Y \geq z) \\
&= 1 - \left(\int_z^\infty \lambda_1 e^{-\lambda_1 x}\, dx\right)\left(\int_z^\infty \lambda_2 e^{-\lambda_2 y}\, dy\right) \\
&= 1 - \left[-e^{-\lambda_1 x}\right]_z^\infty \left[-e^{-\lambda_2 y}\right]_z^\infty \\
&= 1 - e^{-\lambda_1 z - \lambda_2 z}.
\end{aligned}$$

   Differentiating, we get

$$f_Z(z) = \begin{cases} 0 & \text{if } z < 0 \\ (\lambda_1 + \lambda_2)e^{-(\lambda_1+\lambda_2)z} & \text{if } z \geq 0, \end{cases}$$

   which is exactly the density function of the exponential random variable with parameter $\lambda_1 + \lambda_2$.

   (b) By using the standard interpretation of the exponential random variable, convince yourself that it is no surprise that if $X \sim \text{exponential}(\lambda_1)$ and $Y \sim \text{exponential}(\lambda_2)$, and $X$ and $Y$ are independent, then $\min\{X, Y\} \sim \text{exponential}(\lambda_1 + \lambda_2)$.

   **Solution**: If type one occurrences occur at rate $\lambda_1$ per unit time, and type one occurrences occur (independently) at rate $\lambda_2$ per unit time, then occurrences of *some* type occur at rate $\lambda_1 + \lambda_2$ per unit time. $\min\{X, Y\}$ measures the time until the first occurrence of some type (one or two), so it should be modeled by an exponential with parameter $\lambda_1 + \lambda_2$.

2. Chapter 4, problems 1, 2, 5, 7, 9 — see supplementary solutions file.