Math 30530 — Introduction to Probability

Quiz 5 – Monday December 9, 2013

Solutions

Instructions: This is a closed-book quiz. Please do not use any notes.

1. I toss a fair coin repeatedly until I first toss a Head. Let X be the number of times I have to toss the coin (so the possible values of X are $1, 2, 3, \ldots$). Write down the mass function of X (i.e., for each $k \geq 1$, write down $p_X(k) = \Pr(X = k)$). [Note: "fair" means the coin comes up heads 50% of the time.]

   Solution: $\Pr(X = k) = (1/2)^k$.

2. Compute the transform, or moment generating function, of X (the function $M_X(s) = E(e^{sX})$). [Hint: geometric series $1 + x + x^2 + \ldots = 1/(1 - x)$.]

   Solution:

$$\begin{aligned} M_X(s) &= E(e^{sX}) \\ &= \sum_{k=1}^{\infty} e^{sk} \left(\frac{1}{2}\right)^k \\ &= \sum_{k=1}^{\infty} \left(\frac{e^s}{2}\right)^k \\ &= \frac{e^s/2}{1 - e^s/2} \\ &= \frac{e^s}{2 - e^s}. \end{aligned}$$

3. Use your expression for $M_X(s)$ to calculate $E(X)$ (and, for a bonus point, $\mathrm{Var}(X)$).

   Solution:

$$M'_X(s) = \frac{(2 - e^s)e^s + e^{2s}(2 - e^s)}{(2 - e^s)^2},$$

so

$$E(X) = M'_X(0) = \frac{(2 - 1) + 1(2 - 1)}{(2 - 1)^2} = 2.$$

Similarly, one can calculate $E(X^2) = M''_X(0) = 6$ to get $\mathrm{Var}(X) = 6 - 2^2 = 2$.