Bayes' Formula and examples

Math 30530, Fall 2013

September 8, 2013

Example: Who did I beat?

When I play chess, I play Alice (10% of time), Bob (40% of time), and Carole (50% of time). I beat Alice with probability .2, Bob with probability .3, and Carole with probability .4.
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Know:

- $\Omega = A \cup B \cup C$, $\Pr(A) = .1$, $\Pr(B) = .4$, $\Pr(C) = .5$.
- $\Pr(W|A) = .2$, $\Pr(W|B) = .3$, $\Pr(W|C) = .4$

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Want:

$$\begin{aligned}
\Pr(A|W) &= \frac{\Pr(A \cap W)}{\Pr(W)} \\
&= \frac{\Pr(W|A)\Pr(A)}{\Pr(W|A)\Pr(A) + \Pr(W|B)\Pr(B) + \Pr(W|C)\Pr(C)} \\
&= \frac{(.2)(.1)}{(.2)(.1) + (.3)(.4) + (.4)(.5)} \approx .0588.
\end{aligned}$$

Bayes' Formula

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Event B

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$$\begin{aligned}\Pr(A_1|B) &= \frac{\Pr(A_1 \cap B)}{\Pr(B)} \\ &= \frac{\Pr(B|A_1)\Pr(A_1)}{\Pr(B|A_1)\Pr(A_1) + \ldots + \Pr(B|A_n)\Pr(A_n)}\end{aligned}$$

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Terminology:

- $\Pr(A_1)$: **prior** probability of A_1
- $\Pr(A_1|B)$: **posterior** probability of A_1

Example: “Long-haired freaky people need not apply”

Notre Dame campus has 55% men and 45% women. Two-thirds of the women wear their hair long, $1/3$ short. 10% of the men have long hair, 90% short. I see a (random) student from a distance; I can’t make out is it a man or a woman; just that (s)he has long hair. How likely is it that this student is a woman?

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- $\Pr(L|M) = .1$, $\Pr(L|W) = .66$

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Posterior calculation:

$$\begin{aligned}
\Pr(W|L) &= \frac{\Pr(W \cap L)}{\Pr(L)} \\
&= \frac{\Pr(L|W)\Pr(W)}{\Pr(L|W)\Pr(W) + \Pr(L|M)\Pr(M)} \\
&= \frac{(.66)(.45)}{(.66)(.45) + (.1)(.55)} \approx .84
\end{aligned}$$

Example: Gulp, I tested positive

2% of the population have condition X. There's a test for X. Used on subjects who have X, it correctly detects X 98% of the time. Used on subjects who do not have X, it correctly detects the absence of X 98% of the time. I take the test, and it comes out positive. Do I have X?

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Why so low? Among 1,000,000 people, 20,000 have X, 19600 test positive; 980,000 don't have X, 19600 test positive — just as many false positives as true, since number who don't have X much larger than number who do

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Accepting the scholars data as valid, what is the new probability that Shakespeare wrote the play, based on this new evidence?

Answer: probably Shakespeare

- $\Omega = S \cup B$, $\Pr(S) = .6$, $\Pr(B) = .4$.
- $\Pr(E|S) = .08$, $\Pr(E|B) = .02$

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This method was used (successfully) by Mosteller and Wallace to asses which of the disputed Federalist papers were written by Madison, and which by Hamilton

- Frederick Mosteller and David L. Wallace, Inference and Disputed Authorship: The Federalist. Addison-Wesley, Reading, Mass., 1964.