The basic rules of counting

Math 30530, Fall 2013

September 13, 2013

Basic counting rule 1 — The sum rule

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  then the total number of outcomes for the experiment is

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- **Sum rule 2**: if an experiment can proceed in **one of *m* ways**, with
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Convention: $0! = 1$

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Basic counting rule 3 — The overcount rule

- If x is an initial count of some set of objects, and each object you want to count appears y times in x, then the correct count is x/y

Some examples

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$$11!/(4!4!2!1!) = 34,650$$

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  - Homework: $\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$

Splitting a set up into classes of given sizes

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$$\binom{33}{3, 3, \ldots, 3}/11! = \frac{33!}{11!3!^{11}} \approx 5.99 \times 10^{20}$$

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$$\frac{\text{ways of choosing 23 different dates, in order}}{\text{ways of choosing 23 dates, in order}}$$
$$= \frac{365 \times 364 \times \ldots \times 343}{365 \times 365 \times \ldots \times 365}$$
$$\approx .4927 < 50\%$$

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Example: Select 8 single-digit primes, no particular order?

$$\binom{11}{8} = 165$$

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  $$\binom{87}{36} \Big/ \binom{88}{36} \approx .59$$

Summary of counting problems

Sum rule: A OR B? Add

Product rule: A THEN B? Multiply

Overcount rule: Each item counted too many times? Divide

Arranging n items in order: $n!$

Selecting k items from n, WITHOUT REPLACEMENT

- ORDER MATTERS: $\frac{n!}{(n-k)!}$
- ORDER DOESN'T MATTER: $\binom{n}{k} = \frac{n!}{k!(n-k)!}$

Selecting k items from n, WITH REPLACEMENT

- ORDER MATTERS: n^k
- ORDER DOESN'T MATTER: $\binom{n+k-1}{k}$

Partitioning n into classes of size $n_1, n_2, \ldots, n_r$, OR arranging n items in a row when there are n_1 of first type, n_2 of second, etc., and we can't tell the difference within types: $\binom{n}{n_1, n_2, \ldots, n_r} = \frac{n!}{n_1! n_2! \ldots n_r!}$