The basic rules of counting

Math 30530, Fall 2013

September 13, 2013
Basic counting rule 1 — The sum rule

- **Sum rule 1**: if an experiment can proceed in **one of two ways**, with
  - $n_1$ outcomes for the first way, and
  - $n_2$ outcomes for the second,

  then the total number of outcomes for the experiment is

  $$n_1 + n_2$$
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- **Sum rule 2**: if an experiment can proceed in **one of $m$ ways**, with
  - $n_1$ outcomes for the first way,
  - $n_2$ outcomes for the second, ..., and
  - $n_m$ outcomes for the $m$th,

  then the total number of outcomes for the experiment is

  $$n_1 + n_2 + \ldots + n_m$$
Basic counting rule 2 — The product rule

- **Product rule 1**: if an experiment is performed in two stages, with
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  - $n_1$ outcomes for the first stage, and
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  - $n_m$ outcomes for the $m$th, **REGARDLESS OF ALL PREVIOUS**,

  then the total number of outcomes for the experiment is

  $$n_1 n_2 \ldots n_m$$
Arranging objects in a row

In how many ways can \( n \) different, distinct objects be lined up in a row?
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In how many ways can $n$ different, distinct objects be lined up in a row? $n$ options for first item, then $n - 1$ for second (regardless of what was chosen first), then $n - 2$ for second, etc. So by product rule, final count is

$$n \times (n - 1) \times (n - 2) \times \ldots \times 3 \times 2 \times 1$$
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**Example:** Finishing orders in race with 8 runners, no ties allowed?

$$8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 40,320$$
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**Notation:** “\( n \) factorial”

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n! = n \times (n - 1) \times (n - 2) \times \ldots \times 3 \times 2 \times 1
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\( 1! = 1, \ 2! = 2, \ 3! = 6, \ 4! = 24, \ldots, \)
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1! = 1, 2! = 2, 3! = 6, 4! = 24, \ldots,

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34! \approx 29,500,000,000,000,000,000,000,000,000,000,000,000
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**Convention:** \( 0! = 1 \)
Selecting $k$ things from $n$, WITHOUT REPLACEMENT

In how many ways can we pull out $k$ distinct items from among $n$ different, distinct objects?

Order matters: (the $k$ items have to be lined up in a row)

$\frac{n(n-1)\cdots(n-k+1)}{k!} = \binom{n}{k}$ (sometimes $nP_k$)

Example: 1st, 2nd and 3th in race with 8 runners? $8 \times 7 \times 6 = 336$

Order doesn't matter: (the $k$ items are thrown together in a bag)

$n(n-1)\cdots(n-k+1) = \binom{n}{k}$

Example: Top three in race with eight runners? $\binom{8}{3}$
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**Basic counting rule 3 — The overcount rule**

- If \( x \) is an initial count of some set of objects, and each object you want to count appears \( y \) times in \( x \), then the correct count is \( x/y \)
Some examples

- How many ways can 10 people form a committee of 6, with a chair, a secretary, a treasurer and 3 general members?

\[ \binom{10}{6} \cdot \binom{6}{5} \cdot \binom{5}{4} = 25,200 \]

- What if John and Pat will not serve together, and Pat will only serve if Ellen is the chair?

First consider committees with John (so without Pat), then those with Pat (so also with Ellen as chair, and without John), and then those without both John and Pat.

\[ \binom{8}{5} \cdot \binom{6}{5} \cdot \binom{5}{4} + \binom{7}{4} \cdot \binom{5}{4} + \binom{8}{6} \cdot \binom{6}{5} \cdot \binom{5}{4} = 10,780 \]

- How many anagrams of MISSISSPPI?

\[ \frac{11!}{4! \cdot 4! \cdot 2!} = 34,650 \]
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\binom{10}{6} \cdot 6 \cdot 5 \cdot 4 = 25,200
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\binom{8}{5} \cdot 6 \cdot 5 \cdot 4 + \binom{7}{4} \cdot 6 \cdot 5 \cdot 4 + \binom{8}{6} \cdot 6 \cdot 5 \cdot 4 = 10,780
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11!/(4!4!2!1!) = 34,650
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Some properties of binomial coefficients

- \( \binom{n}{n} = 1 \) (justifies \( 0! = 1 \)) and \( \binom{n}{0} = 1 \)
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- \( {n \choose k} = {n \choose n-k} \)
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- Committee/Chair: \( n \binom{n-1}{k-1} = k \binom{n}{k} \)
- Pascal: \( \binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k} \)
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- \( \binom{n}{n} = 1 \) (justifies \( 0! = 1 \)) and \( \binom{n}{0} = 1 \)
- \( \binom{n}{k} = \binom{n}{n-k} \)
- \( (x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} \) (Binomial Theorem)
- Special case: \( \sum_{k=0}^{n} \binom{n}{k} = 2^n \)
- If I repeat the same trial \( n \) times, independently, and each time I have probability \( p \) of success, then the probability that I have **exactly** \( k \) successes is \( \binom{n}{k} p^k (1 - p)^{n-k} \)

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- Pascal: \( \binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k} \)
- Homework: \( \sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n} \)
Splitting a set up into classes of given sizes

In how many ways can we split (partition) a set of size $n$ into $r$ parts, with the first part having size $n_1$, the second size $n_2$, . . . , the $r$th size $n_r$ (and $n = n_1 + \ldots + n_r$)?
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- $r = 2$: same as choosing subset of size $n_1$ (what’s left forms second part of size $n_2$), so $\binom{n}{n_1}$
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- **General $r$:** Two expressions

\[
\binom{n}{n_1} \binom{n-n_1}{n_2} \ldots \binom{n-n_1-\ldots-n_{r-1}}{n_r} \frac{n!}{n_1!n_2!\ldots n_r!} = \binom{n}{n_1, n_2, \ldots, n_r}
\]

Same as number of anagrams of $n$-letter word with $n_1$ repeats of first letter, $n_2$ of second, etc.!
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- **General** $r$: Two expressions

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Same as number of anagrams of $n$-letter word with $n_1$ repeats of first letter, $n_2$ of second, etc.!

**Example**: How many ways to break the class (of 33) into 11 groups of 3?
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**Example:** How many ways to break the class (of 33) into 11 groups of 3?

\[
\binom{33}{3, 3, \ldots, 3}/11! = \frac{33!}{11!3!^{11}} \approx 5.99 \times 10^{20}
\]
Selecting $k$ items from $n$, WITH REPLACEMENT

In how many ways can we pull out $k$ items from among $n$ different, distinct objects, if each time we pull out an item, we note it and put it back?

Order matters: (we produce a list (first item, second, \ldots, last))

Example: 8 digit numbers with prime digits? 4

Example: In a group of 23 people, how likely is it that they all have their birthdays on a different date?

$\dfrac{365 \times 364 \times \ldots \times 343}{365 \times 365 \times \ldots \times 365} \approx 0.4927 < 50\%$
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\[
\mathcal{O} = n^k
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**Example**: In a group of 23 people, how likely is it that they all have their birthdays on a different date?

\[
\frac{\text{ways of choosing 23 different dates, in order}}{\text{ways of choosing 23 dates, in order}} = \frac{365 \times 364 \times \cdots \times 343}{365 \times 365 \times \cdots \times 365} \\
\approx 0.4927 < 50\%
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Selecting $k$ items from $n$, WITH REPLACEMENT

In how many ways can we pull out $k$ distinct items from among $n$ different, distinct objects?

Example: Select 8 single-digit primes, no particular order?\[\binom{11}{8} = 165\]
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In how many ways can we pull out $k$ distinct items from among $n$ different, distinct objects?

- **Order doesn’t matter**: (we just note how many of the first item, how many of the second, etc.)

\[
\binom{n + k - 1}{k}
\]

**Example**: Select 8 single-digit primes, no particular order?

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\binom{11}{8} = 165
\]
Selecting \( k \) items from \( n \), WITH REPLACEMENT

In how many ways can we pull out \( k \) distinct items from among \( n \) different, distinct objects?

- **Order doesn’t matter**: (we just note how many of the first item, how many of the second, etc.)

\[
\binom{n + k - 1}{k}
\]

**Example**: Select 8 single-digit primes, no particular order?

\[
\binom{11}{8} = 165
\]
Some examples

- I have 36 identical prizes to distribute to the class (53 people). All I care about is how many prizes each student gets. How many possible ways to distribute are there?
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$\binom{53 + 36 - 1}{36} = \binom{88}{36} \approx 6 \times 10^{34}$

What’s the probability that Zeke doesn’t get a prize, assuming all ways of distribution equally likely?

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Summary of counting problems

Sum rule: $A \text{ OR } B$? Add

Product rule: $A \text{ THEN } B$? Multiply

Overcount rule: Each item counted too many times? Divide

Arranging $n$ items in order: $n!$

Selecting $k$ items from $n$, WITHOUT REPLACEMENT

- ORDER MATTERS: \( \frac{n!}{(n-k)!} \)
- ORDER DOESN’T MATTER: \( \binom{n}{k} = \frac{n!}{k!(n-k)!} \)

Selecting $k$ items from $n$, WITH REPLACEMENT

- ORDER MATTERS: \( n^k \)
- ORDER DOESN’T MATTER: \( \binom{n+k-1}{k} \)

Partitioning $n$ into classes of size $n_1$, $n_2$, ..., $n_r$, OR arranging $n$ items in a row when there are $n_1$ of first type, $n_2$ of second, etc., and we can’t tell the difference within types: \( \binom{n}{n_1,n_2,...,n_r} = \frac{n!}{n_1!n_2!...n_r!} \)