This is recognized as the exponential CDF with parameter $\lambda + \mu$. Thus, the minimum of two independent exponentials with parameters λ and μ is an exponential with parameter $\lambda + \mu$.

Solution to Problem 4.17. Because the covariance remains unchanged when we add a constant to a random variable, we can assume without loss of generality that X and Y have zero mean. We then have

$$\text{cov}(X - Y, X + Y) = \mathbf{E}\big[(X - Y)(X + Y)\big] = \mathbf{E}[X^2] - \mathbf{E}[Y^2] = \text{var}(X) - \text{var}(Y) = 0,$$

since X and Y were assumed to have the same variance.

Solution to Problem 4.18. We have

$$\text{cov}(R, S) = \mathbf{E}[RS] - \mathbf{E}[R]\mathbf{E}[S] = \mathbf{E}[WX + WY + X^2 + XY] = \mathbf{E}[X^2] = 1,$$

and

$$\text{var}(R) = \text{var}(S) = 2,$$

so

$$\rho(R, S) = \frac{\text{cov}(R, S)}{\sqrt{\text{var}(R)\text{var}(S)}} = \frac{1}{2}.$$

We also have

$$\text{cov}(R, T) = \mathbf{E}[RT] - \mathbf{E}[R]\mathbf{E}[T] = \mathbf{E}[WY + WZ + XY + XZ] = 0,$$

so that

$$\rho(R, T) = 0.$$

Solution to Problem 4.19. To compute the correlation coefficient

$$\rho(X, Y) = \frac{\text{cov}(X, Y)}{\sigma_X \sigma_Y},$$

we first compute the covariance:

$$\begin{aligned}
\text{cov}(X, Y) &= \mathbf{E}[XY] - \mathbf{E}[X]\mathbf{E}[Y] \\
&= \mathbf{E}[aX + bX^2 + cX^3] - \mathbf{E}[X]\mathbf{E}[Y] \\
&= a\mathbf{E}[X] + b\mathbf{E}[X^2] + c\mathbf{E}[X^3] \\
&= b.
\end{aligned}$$

We also have

$$\begin{aligned}
\text{var}(Y) &= \text{var}(a + bX + cX^2) \\
&= \mathbf{E}\big[(a + bX + cX^2)^2\big] - \big(\mathbf{E}[a + bX + cX^2]\big)^2 \\
&= (a^2 + 2ac + b^2 + 3c^2) - (a^2 + c^2 + 2ac) \\
&= b^2 + 2c^2,
\end{aligned}$$

and therefore, using the fact $\mathrm{var}(X) = 1$,

$$\rho(X, Y) = \frac{b}{\sqrt{b^2 + 2c^2}}.$$

Solution to Problem 4.22. If the gambler's fortune at the beginning of a round is a, the gambler bets $a(2p-1)$. He therefore gains $a(2p-1)$ with probability p, and loses $a(2p-1)$ with probability $1-p$. Thus, his expected fortune at the end of a round is

$$a\big(1 + p(2p-1) - (1-p)(2p-1)\big) = a\big(1 + (2p-1)^2\big).$$

Let X_k be the fortune after the kth round. Using the preceding calculation, we have

$$\mathbf{E}[X_{k+1} \,|\, X_k] = \big(1 + (2p-1)^2\big)X_k.$$

Using the law of iterated expectations, we obtain

$$\mathbf{E}[X_{k+1}] = \big(1 + (2p-1)^2\big)\mathbf{E}[X_k],$$

and

$$\mathbf{E}[X_1] = \big(1 + (2p-1)^2\big)x.$$

We conclude that

$$\mathbf{E}[X_n] = \big(1 + (2p-1)^2\big)^n x.$$

Solution to Problem 4.23. (a) Let W be the number of hours that Nat waits. We have

$$\mathbf{E}[X] = \mathbf{P}(0 \le X \le 1)\mathbf{E}[W \,|\, 0 \le X \le 1] + \mathbf{P}(X > 1)\mathbf{E}[W \,|\, X > 1].$$

Since $W > 0$ only if $X > 1$, we have

$$\mathbf{E}[W] = \mathbf{P}(X > 1)\mathbf{E}[W \,|\, X > 1] = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}.$$

(b) Let D be the duration of a date. We have $\mathbf{E}[D \,|\, 0 \le X \le 1] = 3$. Furthermore, when $X > 1$, the conditional expectation of D given X is $(3-X)/2$. Hence, using the law of iterated expectations,

$$\mathbf{E}[D \,|\, X > 1] = \mathbf{E}\big[\mathbf{E}[D \,|\, X] \ \,|\, X > 1\big] = \mathbf{E}\left[\frac{3-X}{2} \ \Big| \ X > 1\right].$$

Therefore,

$$\begin{aligned}
\mathbf{E}[D] &= \mathbf{P}(0 \le X \le 1)\mathbf{E}[D \,|\, 0 \le X \le 1] + \mathbf{P}(X > 1)\mathbf{E}[D \,|\, X > 1] \\
&= \frac{1}{2} \cdot 3 + \frac{1}{2} \cdot \mathbf{E}\left[\frac{3-X}{2} \ \Big| \ X > 1\right] \\
&= \frac{3}{2} + \frac{1}{2}\left(\frac{3}{2} - \frac{\mathbf{E}[X \,|\, X > 1]}{2}\right) \\
&= \frac{3}{2} + \frac{1}{2}\left(\frac{3}{2} - \frac{3/2}{2}\right) \\
&= \frac{15}{8}.
\end{aligned}$$