We have

$$\mathbf{P}(Y_1 = 1 \,|\, X_1 = 1,\, A = a) = \frac{a-1}{2m-1}, \qquad \mathbf{P}(X_1 = 1 \,|\, A = a) = \frac{a}{2m}.$$

Thus

$$\mathbf{E}[S \,|\, A = a] = m\,\frac{a-1}{2m-1} \cdot \frac{a}{2m} = \frac{a(a-1)}{2(2m-1)}.$$

Note that $\mathbf{E}[S \,|\, A = a]$ does not depend on p.

Solution to Problem 2.38. (a) Let X be the number of red lights that Alice encounters. The PMF of X is binomial with $n = 4$ and $p = 1/2$. The mean and the variance of X are $\mathbf{E}[X] = np = 2$ and $\mathrm{var}(X) = np(1-p) = 4 \cdot (1/2) \cdot (1/2) = 1$.

(b) The variance of Alice's commuting time is the same as the variance of the time by which Alice is delayed by the red lights. This is equal to the variance of $2X$, which is $4\mathrm{var}(X) = 4$.

Solution to Problem 2.39. Let X_i be the number of eggs Harry eats on day i. Then, the X_i are independent random variables, uniformly distributed over the set $\{1, \ldots, 6\}$. We have $X = \sum_{i=1}^{10} X_i$, and

$$\mathbf{E}[X] = \mathbf{E}\left(\sum_{i=1}^{10} X_i\right) = \sum_{i=1}^{10} \mathbf{E}[X_i] = 35.$$

Similarly, we have

$$\mathrm{var}(X) = \mathrm{var}\left(\sum_{i=1}^{10} X_i\right) = \sum_{i=1}^{10} \mathrm{var}(X_i),$$

since the X_i are independent. Using the formula of Example 2.6, we have

$$\mathrm{var}(X_i) = \frac{(6-1)(6-1+2)}{12} \approx 2.9167,$$

so that $\mathrm{var}(X) \approx 29.167$.

Solution to Problem 2.40. Associate a success with a paper that receives a grade that has not been received before. Let X_i be the number of papers between the ith success and the $(i+1)$st success. Then we have $X = 1 + \sum_{i=1}^{5} X_i$ and hence

$$\mathbf{E}[X] = 1 + \sum_{i=1}^{5} \mathbf{E}[X_i].$$

After receiving $i-1$ different grades so far ($i-1$ successes), each subsequent paper has probability $(6-i)/6$ of receiving a grade that has not been received before. Therefore, the random variable X_i is geometric with parameter $p_i = (6-i)/6$, so $\mathbf{E}[X_i] = 6/(6-i)$. It follows that

$$\mathbf{E}[X] = 1 + \sum_{i=1}^{5} \frac{6}{6-i} = 1 + 6\sum_{i=1}^{5} \frac{1}{i} = 14.7.$$

Solution to Problem 2.41. (a) The PMF of X is the binomial PMF with parameters $p = 0.02$ and $n = 250$. The mean is $\mathbf{E}[X] = np = 250 \cdot 0.02 = 5$. The desired probability is

$$\mathbf{P}(X = 5) = \binom{250}{5}(0.02)^5(0.98)^{245} = 0.1773.$$

(b) The Poisson approximation has parameter $\lambda = np = 5$, so the probability in (a) is approximated by

$$e^{-\lambda}\frac{\lambda^5}{5!} = 0.1755.$$

(c) Let Y be the amount of money you pay in traffic tickets during the year. Then

$$\mathbf{E}[Y] = \sum_{i=1}^{5} 50 \cdot \mathbf{E}[Y_i],$$

where Y_i is the amount of money you pay on the ith day. The PMF of Y_i is

$$\mathbf{P}(Y_i = y) = \begin{cases} 0.98, & \text{if } y = 0, \\ 0.01, & \text{if } y = 10, \\ 0.006, & \text{if } y = 20, \\ 0.004, & \text{if } y = 50. \end{cases}$$

The mean is

$$\mathbf{E}[Y_i] = 0.01 \cdot 10 + 0.006 \cdot 20 + 0.004 \cdot 50 = 0.42.$$

The variance is

$$\text{var}(Y_i) = \mathbf{E}[Y_i^2] - \big(\mathbf{E}[Y_i]\big)^2 = 0.01 \cdot (10)^2 + 0.006 \cdot (20)^2 + 0.004 \cdot (50)^2 - (0.42)^2 = 13.22.$$

The mean of Y is

$$\mathbf{E}[Y] = 250 \cdot \mathbf{E}[Y_i] = 105,$$

and using the independence of the random variables Y_i, the variance of Y is

$$\text{var}(Y) = 250 \cdot \text{var}(Y_i) = 3,305.$$

(d) The variance of the sample mean is

$$\frac{p(1-p)}{250}$$

so assuming that $|p - \hat{p}|$ is within 5 times the standard deviation, the possible values of p are those that satisfy $p \in [0, 1]$ and

$$(p - 0.02)^2 \leq \frac{25p(1-p)}{250}.$$