CHAPTER 3

Solution to Problem 3.1. The random variable \(Y = g(X)\) is discrete and its PMF is given by
\[
p_Y(1) = P(X \leq 1/3) = 1/3, \quad p_Y(2) = 1 - p_Y(1) = 2/3.
\]

Thus,
\[
E[Y] = \frac{1}{3} \cdot 1 + \frac{2}{3} \cdot 2 = \frac{5}{3}.
\]
The same result is obtained using the expected value rule:
\[
E[Y] = \int_0^1 g(x) f_X(x) \, dx = \int_{1/3}^1 \frac{2}{3} \, dx + \int_{1/3}^1 2 \, dx = \frac{5}{3}.
\]

Solution to Problem 3.2. We have
\[
\int_{-\infty}^{\infty} f_X(x) \, dx = \int_{-\infty}^{\infty} \frac{\lambda}{2} e^{-\lambda|x|} \, dx = 2 \cdot \frac{1}{2} \int_0^{\infty} \lambda e^{-\lambda x} \, dx = 2 \cdot \frac{1}{2} = 1,
\]
where we have used the fact \(\int_0^{\infty} \lambda e^{-\lambda x} \, dx = 1\), i.e., the normalization property of the exponential PDF. By symmetry of the PDF, we have \(E[X] = 0\). We also have
\[
E[X^2] = \int_{-\infty}^{\infty} x^2 \frac{\lambda}{2} e^{-\lambda|x|} \, dx = \int_0^{\infty} x^2 \lambda e^{-\lambda x} \, dx = \frac{2}{\lambda^2},
\]
where we have used the fact that the second moment of the exponential PDF is \(2/\lambda^2\). Thus
\[
\text{var}(X) = E[X^2] - (E[X])^2 = 2/\lambda^2.
\]

Solution to Problem 3.5. Let \(A = bh/2\) be the area of the given triangle, where \(b\) is the length of the base, and \(h\) is the height of the triangle. From the randomly chosen point, draw a line parallel to the base, and let \(A_x\) be the area of the triangle thus formed. The height of this triangle is \(h - x\) and its base has length \(b(h - x)/h\). Thus \(A_x = b(h - x)^2/(2h)\). For \(x \in [0, h]\), we have
\[
F_X(x) = 1 - P(X > x) = 1 - \frac{A_x}{A} = 1 - \frac{b(h - x)^2/(2h)}{bh/2} = 1 - \left(\frac{h - x}{h}\right)^2,
\]
while \(F_X(x) = 0\) for \(x < 0\) and \(F_X(x) = 1\) for \(x > h\).

The PDF is obtained by differentiating the CDF. We have
\[
f_X(x) = \frac{dF_X}{dx}(x) = \begin{cases} \frac{2(h - x)}{h^2}, & \text{if } 0 \leq x \leq h, \\ 0, & \text{otherwise}. \end{cases}
\]
Solution to Problem 3.6. Let $X$ be the waiting time and $Y$ be the number of customers found. For $x < 0$, we have $F_X(x) = 0$, while for $x \geq 0$,
\[
F_X(x) = P(X \leq x) = \frac{1}{2}P(X \leq x \mid Y = 0) + \frac{1}{2}P(X \leq x \mid Y = 1).
\]
Since
\[
P(X \leq x \mid Y = 0) = 1,
\]
\[
P(X \leq x \mid Y = 1) = 1 - e^{-\lambda x},
\]
we obtain
\[
F_X(x) = \begin{cases} 
\frac{1}{2}(2 - e^{-\lambda x}), & \text{if } x \geq 0, \\
0, & \text{otherwise}.
\end{cases}
\]
Note that the CDF has a discontinuity at $x = 0$. The random variable $X$ is neither discrete nor continuous.

Solution to Problem 3.7. (a) We first calculate the CDF of $X$. For $x \in [0, r]$, we have
\[
F_X(x) = P(X \leq x) = \frac{\pi x^2}{\pi r^2} = \left(\frac{x}{r}\right)^2.
\]
For $x < 0$, we have $F_X(x) = 0$, and for $x > r$, we have $F_X(x) = 1$. By differentiating, we obtain the PDF
\[
f_X(x) = \begin{cases} 
\frac{2x}{r^2}, & \text{if } 0 \leq x \leq r, \\
0, & \text{otherwise}.
\end{cases}
\]
We have
\[
E[X] = \int_0^r \frac{2x^2}{r^2} dx = \frac{2r}{3}.
\]
Also
\[
E[X^2] = \int_0^r \frac{2x^3}{r^2} dx = \frac{r^2}{2},
\]
so
\[
\text{var}(X) = E[X^2] - (E[X])^2 = \frac{r^2}{2} - \frac{4r^2}{9} = \frac{r^2}{18}.
\]
(b) Alvin gets a positive score in the range $[1/t, \infty)$ if and only if $X \leq t$, and otherwise he gets a score of 0. Thus, for $s < 0$, the CDF of $S$ is $F_S(s) = 0$. For $0 \leq s < 1/t$, we have
\[
F_S(s) = P(S \leq s) = P(\text{Alvin’s hit is outside the inner circle}) = 1 - P(X \leq t) = 1 - \frac{t^2}{r^2}.
\]
For $1/t < s$, the CDF of $S$ is given by
\[
F_S(s) = P(S \leq s) = P(X \leq t)P(S \leq s \mid X \leq t) + P(X > t)P(S \leq s \mid X > t).
\]
We have
\[ P(X \leq t) = \frac{t^2}{r^2}, \quad P(X > t) = 1 - \frac{t^2}{r^2}, \]
and since \( S = 0 \) when \( X > t \),
\[ P(S \leq s \mid X > t) = 1. \]
Furthermore,
\[ P(S \leq s \mid X \leq t) = P(1/X \leq s \mid X \leq t) = P \left( 1 + \frac{1}{s} \leq X \leq t \right) = \frac{\pi t^2 - \pi (1/s)^2}{\pi r^2} = 1 - \frac{1}{s^2 t^2}. \]
Combining the above equations, we obtain
\[ P(S \leq s) = \frac{t^2}{r^2} \left( 1 - \frac{1}{s^2 t^2} \right) + 1 - \frac{t^2}{r^2} = 1 - \frac{1}{s^2 t^2}. \]
Collecting the results of the preceding calculations, the CDF of \( S \) is
\[ F_S(s) = \begin{cases} 
0, & \text{if } s < 0, \\
1 - \frac{t^2}{r^2}, & \text{if } 0 \leq s < 1/t, \\
1 - \frac{1}{s^2 t^2}, & \text{if } 1/t \leq s.
\end{cases} \]
Because \( F_S \) has a discontinuity at \( s = 0 \), the random variable \( S \) is not continuous.

**Solution to Problem 3.8.** (a) By the total probability theorem, we have
\[ F_X(x) = P(X \leq x) = pP(Y \leq x) + (1-p)P(Z \leq x) = pF_Y(x) + (1-p)F_Z(x). \]
By differentiating, we obtain
\[ f_X(x) = pf_Y(x) + (1-p)f_Z(x). \]
(b) Consider the random variable \( Y \) that has PDF
\[ f_Y(y) = \begin{cases} 
\lambda e^{\lambda y}, & \text{if } y < 0 \\
0, & \text{otherwise},
\end{cases} \]
and the random variable \( Z \) that has PDF
\[ f_Z(z) = \begin{cases} 
\lambda e^{-\lambda z}, & \text{if } y \geq 0 \\
0, & \text{otherwise}.
\end{cases} \]
We note that the random variables \(-Y\) and \( Z\) are exponential. Using the CDF of the exponential random variable, we see that the CDFs of \( Y \) and \( Z \) are given by
\[ F_Y(y) = \begin{cases} 
e^{\lambda y}, & \text{if } y < 0, \\
1, & \text{if } y \geq 0,
\end{cases} \]
\[ F_Z(z) = \begin{cases} 0, & \text{if } z < 0, \\ 1 - e^{-\lambda z}, & \text{if } z \geq 0. \end{cases} \]

We have \( f_X(x) = pf_Y(x) + (1 - p)f_Z(x) \), and consequently \( F_X(x) = pF_Y(x) + (1 - p)F_Z(x) \). It follows that
\[
F_X(x) = \begin{cases} pe^{\lambda x}, & \text{if } x < 0, \\ p + (1 - p)(1 - e^{-\lambda x}), & \text{if } x \geq 0. \end{cases}
\]

**Solution to Problem 3.11.**

(a) \( X \) is a standard normal, so by using the normal table, we have \( P(X \leq 1.5) = \Phi(1.5) = 0.9332 \). Also \( P(X \leq -1) = 1 - \Phi(1) = 1 - 0.8413 = 0.1587 \).

(b) The random variable \( (Y - 1)/2 \) is obtained by subtracting from \( Y \) its mean (which is 1) and dividing by the standard deviation (which is 2), so the PDF of \( (Y - 1)/2 \) is the standard normal.

(c) We have, using the normal table,
\[
P(-1 \leq Y \leq 1) = P\left(-1 \leq \frac{(Y - 1)}{2} \leq 0\right) = P(-1 \leq Z \leq 0) = P(0 \leq Z \leq 1) = \Phi(1) - \Phi(0) = 0.8413 - 0.5 = 0.3413,
\]
where \( Z \) is a standard normal random variable.

**Solution to Problem 3.12.** The random variable \( Z = X/\sigma \) is a standard normal, so
\[
P(X \geq k\sigma) = P(Z \geq k) = 1 - \Phi(k).
\]

From the normal tables we have
\[
\Phi(1) = 0.8413, \quad \Phi(2) = 0.9772, \quad \Phi(3) = 0.9986.
\]

Thus \( P(X \geq \sigma) = 0.1587, \ P(X \geq 2\sigma) = 0.0228, \ P(X \geq 3\sigma) = 0.0014. \)

We also have
\[
P(|X| \leq k\sigma) = P(|Z| \leq k) = \Phi(k) - P(Z \leq -k) = \Phi(k) - (1 - \Phi(k)) = 2\Phi(k) - 1.
\]

Using the normal table values above, we obtain
\[
P(|X| \leq \sigma) = 0.6826, \quad P(|X| \leq 2\sigma) = 0.9544, \quad P(|X| \leq 3\sigma) = 0.9972,
\]
where \( t \) is a standard normal random variable.