Solution to Problem 4.1. Let $Y = \sqrt{|X|}$. We have, for $0 \leq y \leq 1$,

$$F_Y(y) = P(Y \leq y) = P(\sqrt{|X|} \leq y) = P(-y^2 \leq X \leq y^2) = y^2,$$

and therefore by differentiation,

$$f_Y(y) = 2y, \quad \text{for } 0 \leq y \leq 1.$$

Let $Y = -\ln|X|$. We have, for $y \geq 0$,

$$F_Y(y) = P(Y \leq y) = P(\ln|X| \geq -y) = P(X \geq e^{-y}) + P(X \leq -e^{-y}) = 1 - e^{-y},$$

and therefore by differentiation

$$f_Y(y) = e^{-y}, \quad \text{for } y \geq 0,$$

so $Y$ is an exponential random variable with parameter 1. This exercise provides a method for simulating an exponential random variable using a sample of a uniform random variable.

Solution to Problem 4.2. Let $Y = e^X$. We first find the CDF of $Y$, and then take the derivative to find its PDF. We have

$$P(Y \leq y) = P(e^X \leq y) = \begin{cases} P(X \leq \ln y), & \text{if } y > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Therefore,

$$f_Y(y) = \begin{cases} \frac{d}{dx} F_X(\ln y), & \text{if } y > 0, \\ 0, & \text{otherwise,} \end{cases}$$

$$= \begin{cases} \frac{1}{y} f_X(\ln y), & \text{if } y > 0, \\ 0, & \text{otherwise.} \end{cases}$$

When $X$ is uniform on $[0, 1]$, the answer simplifies to

$$f_Y(y) = \begin{cases} \frac{1}{y}, & \text{if } 0 < y \leq e, \\ 0, & \text{otherwise.} \end{cases}$$

Solution to Problem 4.3. Let $Y = |X|^{1/3}$. We have

$$F_Y(y) = P(Y \leq y) = P(|X|^{1/3} \leq y) = P(-y^3 \leq X \leq y^3) = F_X(y^3) - F_X(-y^3),$$
and therefore, by differentiating,
\[ f_Y(y) = 3y^2 f_X(y^4) + 3y^2 f_X(-y^4), \quad \text{for } y > 0. \]

Let \( Y = |X|^{1/4} \). We have
\[ F_Y(y) = P(Y \leq y) = P(|X|^{1/4} \leq y) = P(-y^4 \leq X \leq y^4) = F_X(y^4) - F_X(-y^4), \]
and therefore, by differentiating,
\[ f_Y(y) = 4y^3 f_X(y^4) + 4y^3 f_X(-y^4), \quad \text{for } y > 0. \]

**Solution to Problem 4.4.** We have
\[
F_Y(y) = \begin{cases} 
0, & \text{if } y \leq 0, \\
F_X(5) - F_X(5 - y), & \text{if } 0 \leq y \leq 5, \\
F_X(20) - F_X(20 - y), & \text{if } 5 < y \leq 15, \\
1, & \text{if } y > 15.
\end{cases}
\]

Using the CDF of \( X \), we have
\[ P(5 - y \leq X \leq 5) = F_X(5) - F_X(5 - y), \]
\[ P(20 - y \leq X \leq 20) = F_X(20) - F_X(20 - y). \]

Thus,
\[
F_Y(y) = \begin{cases} 
0, & \text{if } y \leq 0, \\
F_X(5) - F_X(5 - y) + F_X(20) - F_X(20 - y), & \text{if } 0 \leq y \leq 5, \\
F_X(20) - F_X(20 - y), & \text{if } 5 < y \leq 15, \\
1, & \text{if } y > 15.
\end{cases}
\]

Differentiating, we obtain
\[
f_Y(y) = \begin{cases} 
f_X(5 - y) + f_X(20 - y), & \text{if } 0 \leq y \leq 5, \\
f_X(20 - y), & \text{if } 5 < y \leq 15, \\
0, & \text{otherwise},
\end{cases}
\]
consistent with the result of Example 3.14.

**Solution to Problem 4.5.** Let \( Z = |X - Y| \). We have
\[ F_Z(z) = P(|X - Y| \leq z) = 1 - (1 - z)^2. \]

(To see this, draw the event of interest as a subset of the unit square and calculate its area.) Taking derivatives, the desired PDF is
\[ f_Z(z) = \begin{cases} 
2(1 - z), & \text{if } 0 \leq z \leq 1, \\
0, & \text{otherwise},
\end{cases} \]
Solution to Problem 4.6. Let $Z = |X - Y|$. To find the CDF, we integrate the joint PDF of $X$ and $Y$ over the region where $|X - Y| \leq z$ for a given $z$. In the case where $z \leq 0$ or $z \geq 1$, the CDF is 0 and 1, respectively. In the case where $0 < z < 1$, we have

$$F_Z(z) = P(X - Y \leq z, X \geq Y) + P(Y - X \leq z, X < Y).$$

The events $\{X - Y \leq z, X \geq Y\}$ and $\{Y - X \leq z, X < Y\}$ can be identified with subsets of the given triangle. After some calculation using triangle geometry, the areas of these subsets can be verified to be $z/2 + z^2/4$ and $1/4 - (1 - z)^2/4$, respectively.

Therefore, since $f_{X,Y}(x,y) = 1$ for all $(x,y)$ in the given triangle,

$$F_Z(z) = \left(\frac{z}{2} + \frac{z^2}{4}\right) + \left(\frac{1}{4} - \frac{(1-z)^2}{4}\right) = z.$$

Thus,

$$F_Z(z) = \begin{cases} 0, & \text{if } z \leq 0, \\ z, & \text{if } 0 < z < 1, \\ 1, & \text{if } z \geq 1. \end{cases}$$

By taking the derivative with respect to $z$, we obtain

$$f_Z(z) = \begin{cases} 1, & \text{if } 0 \leq z \leq 1, \\ 0, & \text{otherwise}. \end{cases}$$

Solution to Problem 4.7. Let $X$ and $Y$ be the two points, and let $Z = \max\{X, Y\}$. For any $t \in [0,1]$, we have

$$P(Z \leq t) = P(X \leq t)P(Y \leq t) = t^2,$$

and by differentiating, the corresponding PDF is

$$f_Z(z) = \begin{cases} 0, & \text{if } z \leq 0, \\ 2z, & \text{if } 0 \leq z \leq 1, \\ 0, & \text{if } z \geq 1. \end{cases}$$

Thus, we have

$$E[Z] = \int_{-\infty}^{\infty} z f_Z(z)dz = \int_{0}^{1} 2z^2dz = \frac{2}{3}.$$

The distance of the largest of the two points to the right endpoint is $1 - Z$, and its expected value is $1 - E[Z] = 1/3$. A symmetric argument shows that the distance of the smallest of the two points to the left endpoint is also $1/3$. Therefore, the expected distance between the two points must also be $1/3$.

Solution to Problem 4.8. Note that $f_X(x)$ and $f_Y(z-x)$ are nonzero only when $x \geq 0$ and $x \leq z$, respectively. Thus, in the convolution formula, we only need to integrate for $x$ ranging from 0 to $z$:

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x)f_Y(z-x)dx = \int_{0}^{z} \lambda e^{-\lambda x} \lambda e^{-\lambda(z-x)}dx = \lambda^2 e^{-z} \int_{0}^{z} dx = \lambda^2 z e^{-\lambda z}.$$
Solution to Problem 4.9. Let $Z = X - Y$. We will first calculate the CDF $F_Z(z)$ by considering separately the cases $z \geq 0$ and $z < 0$. For $z \geq 0$, we have (see the left side of Fig. 4.6)

$$F_Z(z) = P(X - Y \leq z) = 1 - P(X - Y > z)$$

$$= 1 - \int_0^{\infty} \left( \int_{x+y}^{\infty} f_{X,Y}(x,y) \, dx \right) \, dy$$

$$= 1 - \int_0^{\infty} \mu e^{-\mu y} \left( \int_{x+y}^{\infty} \lambda e^{-\lambda x} \, dx \right) \, dy$$

$$= 1 - \int_0^{\infty} \mu e^{-\mu y} e^{-\lambda(x+y)} \, dy$$

$$= 1 - e^{-\lambda x} \int_0^{\infty} \mu e^{-(\lambda+\mu)y} \, dy$$

$$= 1 - \frac{\mu}{\lambda + \mu} e^{-\lambda x}.$$

For the case $z < 0$, we have using the preceding calculation

$$F_Z(z) = 1 - F_Z(-z) = 1 - \left( 1 - \frac{\lambda}{\lambda + \mu} e^{-\mu(-z)} \right) = \frac{\lambda}{\lambda + \mu} e^{\mu z}.$$

Combining the two cases $z \geq 0$ and $z < 0$, we obtain

$$F_Z(z) = \begin{cases} 
1 - \frac{\mu}{\lambda + \mu} e^{-\lambda x}, & \text{if } z \geq 0, \\
\frac{\lambda}{\lambda + \mu} e^{\mu z}, & \text{if } z < 0.
\end{cases}$$

The PDF of $Z$ is obtained by differentiating its CDF. We have

$$f_Z(x) = \begin{cases} 
\frac{\lambda \mu}{\lambda + \mu} e^{-\lambda x}, & \text{if } z \geq 0, \\
\frac{\lambda \mu}{\lambda + \mu} e^{\mu x}, & \text{if } z < 0.
\end{cases}$$

For an alternative solution, fix some $z \geq 0$ and note that $f_Y(x-z)$ is nonzero only when $x \geq z$. Thus,

$$f_{X-Y}(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(x-z) \, dx$$

$$= \int_{z}^{\infty} \lambda e^{-\lambda x} \mu e^{-\mu(x-z)} \, dx$$

$$= \lambda \mu e^{\lambda z} \int_{z}^{\infty} e^{-(\lambda+\mu)x} \, dx$$

$$= \lambda \mu e^{\lambda z} \frac{1}{\lambda + \mu} e^{-(\lambda+\mu)x}$$

$$= \frac{\lambda \mu}{\lambda + \mu} e^{-\mu z}.$$