Examples involving conditional probability

Math 30530, Fall 2013

September 5, 2013
I’m always late to work

60% of days I walk to work. On those days
  → I’m late 80% of the time
  → I’m on time 20% of the time
40% of days I drive to work. On those days
  → I’m late 50% of the time
  → I’m on time 50% of the time
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**Question 1:** What’s the probability that I’m late for work?

**Answer:**
$$\Pr(L) = \Pr(L \cap W) + \Pr(L \cap D) = \Pr(W) \Pr(L | W) + \Pr(D) \Pr(L | D) = \left(0.6\right)\left(0.8\right) + \left(0.4\right)\left(0.5\right) = 0.68$$

**Question 2:** On a day that I’m late for work, what’s the probability that I drove?

**Answer:**
$$\Pr(D | L) = \frac{\Pr(D \cap L)}{\Pr(L)} = \frac{\Pr(D) \Pr(L | D)}{\Pr(L)} = \frac{\left(0.4\right)\left(0.5\right)}{0.68} \approx 0.29.$$
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Answer: \( \Pr(D|L) = \frac{\Pr(D \cap L)}{\Pr(L)} = \frac{(0.4)(0.5)}{0.68} \approx 0.29. \)
I got a flush!

I draw 5 cards from a well-shuffled deck.
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**Question:** What’s the probability that all 5 cards are hearts?
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**Answer:** Let $A_i$ be the event that the $i$th card drawn is a heart.
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\[
\Pr(A_1 \cap A_2 \cap A_3 \cap A_4 \cap A_5) = \Pr(A_1) \times \Pr(A_2|A_1) \times \Pr(A_3|A_1 \cap A_2) \times \Pr(A_4|A_1 \cap A_2 \cap A_3) \times \Pr(A_5|A_1 \cap A_2 \cap A_3 \cap A_4)
\]

\[
= \left( \frac{13}{52} \right) \left( \frac{12}{51} \right) \left( \frac{11}{50} \right) \left( \frac{10}{49} \right) \left( \frac{9}{48} \right)
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$$

$$
= \frac{13}{52} \times \frac{12}{51} \times \frac{11}{50} \times \frac{10}{49} \times \frac{9}{48}
$$

$$
= \frac{13 \times 12 \times 11 \times 10 \times 9}{52 \times 51 \times 50 \times 49 \times 48}
$$
Am I at the gym?

- If I go to the gym today, there’s an 80% chance that I’ll go tomorrow.
- If I skip the gym today, there’s a 40% chance that I’ll go tomorrow.
- I go to the gym today.

Question: What’s the probability that I go 30 days from now?

Answer: Let \( G_i = \{ \text{I go to the gym on day } i \} \) (today is day 0).

\[
\Pr(G_0) = 1
\]

\[
\Pr(G_1) = 0.8
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\[
\Pr(G_2) = \Pr(G_2 \cap G_1) + \Pr(G_2 \cap G_1^c)
\]
\[
= \Pr(G_1) \Pr(G_2|G_1) + \Pr(G_1^c) \Pr(G_2|G_1^c)
\]
\[
= .8 \Pr(G_1) + .4 (1 - \Pr(G_1))
\]
\[
= .4 \Pr(G_1) + .4 (= .72)
\]
Am I at the gym?

\begin{align*}
\Pr(G_2) &= \Pr(G_2 \cap G_1) + \Pr(G_2 \cap G_1^c) \\
&= \Pr(G_1) \Pr(G_2 | G_1) + \Pr(G_1^c) \Pr(G_2 | G_1^c) \\
&= .8 \Pr(G_1) + .4(1 - \Pr(G_1)) \\
&= .4 \Pr(G_1) + .4 (= .72) \\
\Pr(G_3) &= \Pr(G_2) \Pr(G_3 | G_2) + \Pr(G_2^c) \Pr(G_3 | G_2^c) \\
&= .8 \Pr(G_2) + .4(1 - \Pr(G_2)) \\
&= .4 \Pr(G_2) + .4 (= .688)
\end{align*}
Am I at the gym?

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In general,

\[ \Pr(G_n) = \Pr(G_{n-1}) \Pr(G_n | G_{n-1}) + \Pr(G_{n-1}^c) \Pr(G_n | G_{n-1}^c) \]
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In general,

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\Pr(G_n) &= \Pr(G_{n-1}) \Pr(G_n|G_{n-1}) + \Pr(G_{n-1}^c) \Pr(G_n|G_{n-1}^c) \\
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\end{align*}
\]

\[
\Pr(G_{30}) \approx \frac{2}{3} + 10^{-12}, \text{ and for } n \text{ above about 10, } \Pr(G_n) \text{ basically indistinguishable from } 2/3
\]