Independence of events (with example)
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We use (1) as definition of independent, and say that \( A, B \) are *independent*.

Independence means: nothing you learn about one, tells you anything new about the other. E.g., if \( A, B \) are independent then:

\[ \Pr(B|A) = \Pr(B) \]
\[ \Pr(A|B^c) = \Pr(A) \]
\[ \Pr(A^c|B^c) = \Pr(A^c) \]
Independence of many events

$A_1, A_2, \ldots, A_n$ are independent if (informally) nothing you learn about some of the $A$'s, tells you anything new about another. E.g., if $A_1, \ldots, A_n$, are independent then:

$$\Pr(A_1 | A_2 \cap A_3 \cap A_7) = \Pr(A_1)$$

$$\Pr(A_5^c | A_3 \cap A_7^c \cap A_{11}) = \Pr(A_5^c)$$

$$\Pr(A_6 | A_8) = \Pr(A_6)$$
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\begin{align*}
\Pr(A_1 \mid A_2 \cap A_3 \cap A_7) &= \Pr(A_1) \\
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\text{Probability of intersection} = \text{product of probabilities}.
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I.e., for $A, B, C$, this means:

\[
\Pr(A \cap B) = \Pr(A) \Pr(B), \quad \Pr(A \cap C) = \Pr(A) \Pr(C), \quad \Pr(B \cap C) = \Pr(B) \Pr(C)
\]

AND

\[
\Pr(A \cap B \cap C) = \Pr(A) \Pr(B) \Pr(C).
\]
Example

A bulb will work for one year with probability $p$. To light my basement, I install $n$ bulbs, all operating independently. What is the probability that after one year, at least one of the bulbs will still be working?

Answer

Let $A_i$ be the event that the $i$th bulb works after a year. The $A_i$'s are assumed independent, so

$$\Pr(\geq 1) = 1 - \Pr(0) = 1 - \Pr(A_1^c \cap \ldots \cap A_n^c) = 1 - \Pr(A_1^c) \ldots \Pr(A_n^c) = 1 - (1 - p)^n.$$  

What is the probability that after one year, exactly $k$ of the bulbs will still be working?

$$\Pr(k) = \binom{n}{k} p^k (1 - p)^{n-k}$$

where $\binom{n}{k}$ is the number of ways of selecting $k$ bulbs out of the $n$ to be working.
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