Independence of events (with example)

Math 30530, Fall 2013

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Independence means: nothing you learn about one, tells you anything new about the other. E.g., if *A*, *B* are independent then:

$$\begin{aligned} \Pr(B|A) &= \Pr(B) \\ \Pr(A|B^c) &= \Pr(A) \\ \Pr(A^c|B^c) &= \Pr(A^c) \end{aligned}$$

Independence of many events

$A_1, A_2, \ldots, A_n$ are *independent* if (informally) nothing you learn about some of the A's, tells you anything new about another. E.g., if $A_1, \ldots, A_n$, are independent then:

$$\begin{aligned} \Pr(A_1 | A_2 \cap A_3 \cap A_7) &= \Pr(A_1) \\ \Pr(A_5^c | A_3 \cap A_7^c \cap A_{11}) &= \Pr(A_5^c) \\ \Pr(A_6 | A_8) &= \Pr(A_6) \end{aligned}$$

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Probability of intersection = product of probabilities.

I.e., for A, B, C, this means:

$\Pr(A \cap B) = \Pr(A)\Pr(B)$, $\Pr(A \cap C) = \Pr(A)\Pr(C)$, $\Pr(B \cap C) = \Pr(B)\Pr(C)$

AND

$$\Pr(A \cap B \cap C) = \Pr(A)\Pr(B)\Pr(C).$$

Example

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Answer: Let A_i be the event that the ith bulb works after a year. The A_i's are assumed independent, so

$$\begin{aligned}
\Pr(\geq 1) &= 1 - \Pr(0) \\
&= 1 - \Pr(A_1^c \cap \ldots \cap A_n^c) \\
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$$\Pr(k) = (\#(n,k))\, p^k (1-p)^{n-k}$$

where $\#(n,k)$ is the number of ways of selecting k bulbs out of the n to be working.