Weak Law of Large Numbers & Central Limit Theorem

Math 30530, Fall 2013

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Weak Law of Large Numbers

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$$\Pr(|M_n - \mu| \geq \varepsilon) \to 0 \text{ as } n \to \infty$$
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**Interpretation**: Repeat an experiment many times independently, record the answers, and average them. By making “many” large enough, we can be sure that the average of the readings is arbitrarily close to the theoretical experiment average, with arbitrarily high probability (arbitrarily close to 1)
$X_1, X_2, X_3, \ldots, X_n$ are independent copies of the same random variable, all with mean $\mu$, variance $\sigma^2$, $M_n = \frac{X_1+X_2+\ldots+X_n}{n}$
Effective version

\( X_1, X_2, X_3, \ldots, X_n \) are independent copies of the same random variable, all with mean \( \mu \), variance \( \sigma^2 \), \( M_n = \frac{X_1 + X_2 + \ldots + X_n}{n} \)

**Weak law of large numbers:** For every \( \varepsilon > 0 \),

\[
\Pr(\left| M_n - \mu \right| \geq \varepsilon) \leq \frac{\sigma^2}{n\varepsilon^2}
\]
Effective version

\(X_1, X_2, X_3, \ldots, X_n\) are independent copies of the same random variable, all with mean \(\mu\), variance \(\sigma^2\),

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\[\Pr(|M_n - \mu| \geq \varepsilon) \leq \frac{\sigma^2}{n\varepsilon^2}\]

**Example:** I roll a die 1,000 times. How sure can I be that the average of the rolls is between 3 and 4?

\[\text{Solution: Here } X_i \text{ is result of roll of die, } \mu = 3.5, \sigma^2 = 2.92, n = 1000, \varepsilon = 0.5, \text{ and } \Pr(|M_{1000} - 3.5| \geq 0.5) \leq \frac{2.92}{1000(0.5)^2} = 0.01168, \text{ so I can be around 98.8% sure of an average between 3 and 4.}\]
$X_1, X_2, X_3, \ldots, X_n$ are independent copies of the same random variable, all with mean $\mu$, variance $\sigma^2$, $M_n = \frac{X_1 + X_2 + \ldots + X_n}{n}$

**Weak law of large numbers**: For every $\varepsilon > 0$, 

$$\Pr(|M_n - \mu| \geq \varepsilon) \leq \frac{\sigma^2}{n\varepsilon^2}$$

**Example**: I roll $s$ dice 1,000 times. How sure can I be that the average of the rolls is between 3 and 4?

**Solution**: Here $X_i$ is result of roll of dice, $\mu = 3.5$, $\sigma^2 = 2.92$, $n = 1000$, $\varepsilon = .5$, and

$$\Pr(|M_{1000} - 3.5| \geq .5) \leq \frac{2.92}{1000(.5)^2} = .01168,$$

so I can be around 98.8% sure of an average between 3 and 4.
More involved example

**Example**: A certain brand of lightbulb has lifetime that is exponentially distributed with mean $A$ hours, $A$ unknown. I try to estimate $A$ by letting $n$ lightbulbs run independently, and recording & averaging their lifetimes. How large should $n$ be, so that I can be at least 90% sure that the estimate I get is within 5% of the actual average $A$?
More involved example

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We’ll use $\Pr(|M_n - \mu| \geq \varepsilon) \leq \sigma^2 / (n\varepsilon^2)$
More involved example

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We’ll use $\Pr(|M_n - \mu| \geq \varepsilon) \leq \sigma^2/(n\varepsilon^2)$

Here $M_n = (X_1 + \ldots + X_n)/n$ with $X_i \sim \text{exponential}(\lambda)$ ($\lambda$ unknown), $\mu = 1/\lambda = A$, $\sigma^2 = 1/\lambda^2 = A^2$, $\varepsilon = .05\mu = .05A$, so

$$\Pr(|M_n - A| \geq .05A) \leq \frac{A^2}{n(.05A)^2} = \frac{400}{n}$$

Want $n$ large enough so that this probability is at most .1, so $n = 4000$ large enough.
Another example — accumulating rounding error

**Example:** I estimate the sum of \( n \) random real numbers by rounding each to the nearest integer, and adding the resulting integers. What is the probability that the total error is at most \( \pm \sqrt{n} \)?
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**Example:** I estimate the sum of \( n \) random real numbers by rounding each to the nearest integer, and adding the resulting integers. What is the probability that the total error is at most \( \pm \sqrt{n} \)?

Let \( X_i \) be error in \( i \)th number, so \( X_i \sim \text{Uniform}(-1/2, 1/2) \). Let \( T_n \) be the total error, so \( T_n = X_1 + \ldots + X_n \), all \( X_i \) independent. Each \( X_i \) has \( \mu = 0, \sigma^2 = 1/12 \). By weak law of large numbers,

\[
\Pr(|M_n - 0| \geq \varepsilon) \leq \frac{1}{12n\varepsilon^2},
\]

so, since \( T_n = nM_n \), \( \Pr(|T_n| \geq n\varepsilon) \leq \frac{1}{12n\varepsilon^2} \). Want \( \Pr(|T_n| \geq \sqrt{n}) \), so set \( \varepsilon = 1/\sqrt{n} \), to get

\[
\Pr(|T_n| \geq \sqrt{n}) \leq \frac{(\sqrt{n^2})}{12n} = \frac{1}{12}.
\]

Example: with 400 numbers, I can be at least \( 11/12 \) sure that the error I make using rounding is no more than \( \pm 20 \).
Central Limit Theorem

**Informally**: Add together lots of independent copies of a random variable. The result is very close to being a normal random variable.
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**Less informally:** Let $X_1, X_2, \ldots, X_n$ be a collection of independent copies of the same random variable, with $X_i$ having mean $\mu$ and variance $\sigma^2$. Then

$$X_1 + \ldots + X_n \approx \text{Normal}(n\mu, n\sigma^2).$$
Central Limit Theorem

**Informally:** Add together lots of independent copies of a random variable. The result is very close to being a normal random variable.

**Less informally:** Let $X_1, X_2, \ldots, X_n$ be a collection of independent copies of the same random variable, with $X_i$ having mean $\mu$ and variance $\sigma^2$. Then

$$X_1 + \ldots + X_n \approx \text{Normal}(n\mu, n\sigma^2).$$

**Precisely:** Let $X_1, X_2, \ldots, X_n$ be as above. Set

$$S_n = \frac{X_1 + \ldots + X_n - n\mu}{\sqrt{n}\sigma}$$

(so $E(S_n) = 0$, $\text{Var}(S_n) = 1$). Then, for each $x \in \mathbb{R}$,

$$\Pr(S_n \leq x) \to \Pr(Z \leq x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-z^2/2} \, dz$$

as $n \to \infty$. 
Example — accumulating rounding error

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Example — accumulating rounding error

**Example**: I estimate the sum of $n$ random real numbers by rounding each to the nearest integer, and adding the resulting integers. What is the probability that the total error is at most $\pm \sqrt{n}$?

Let $X_i$ be error in $i$th number, so $X_i \sim \text{Uniform}(-1/2, 1/2)$. Let $T_n$ be the total error, so $T_n = X_1 + \ldots + X_n$, all $X_i$ independent. Each $X_i$ has $\mu = 0$, $\sigma^2 = 1/12$. By central limit theorem, $T_n \approx \text{Normal}(0, n/12)$, so

$$
\Pr(|T_n| \leq \sqrt{n}) \approx \Pr(-\sqrt{n} \leq \text{Normal}(0, n/12) \leq \sqrt{n})
= \Pr(-\sqrt{12} \leq Z \leq \sqrt{12})
\approx .99946.
$$

Example: with 400 numbers, I can actually be at least .999 sure that the error I make using rounding is no more than $\pm 20$ (and at least .95 sure of no more than $\pm 11.3$ error).