Math 30530, Introduction to Probability

Spring 2019

Notes for final — solutions to practice questions

1. There are twelve parking spaces in a row outside the main building - reserved (in no particular order) for Provost Burrish and his eleven associate provosts.

   (a) One day, six of the twelve drive to work, and when each one arrives they choose a random empty parking spot to park in. When the sixth person arrives, what is the probability that he finds that the spots at both ends are free?

   (b) On another day, eight of the twelve drive, and again they each choose a random empty parking spot to park in. What is the probability that after all eight have parked, there are four consecutive free parking spaces?

   Solutions:

   (a) Denominator: number of ways 6 spots can be taken, one after another. Numerator: number of ways 6 spots can be taken from the middle ten, one after another. Probability:

$$\frac{10 \times 9 \times 8 \times 7 \times 6}{12 \times 11 \times 10 \times 9 \times 8}$$

   (b) 9 ways to choose the location of 4 consecutive left empty, so (arguing as in (a)), probability

$$9 \times \frac{8!}{12 \times 11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5}$$

2. I buy three bags of holiday themed maltballs from the South Bend Chocolate Company. By the time I get home with them, all that's left in bag I is 3 red maltballs and 2 green ones; all that's left in bag II is 2 red and 1 green, and all that's left in bag III is 1 red and 3 green. I give the three bags to my wife.

   (a) If she selects one maltball from each bag, what is the probability that she selects three red balls?

   (b) If she instead selects one bag at random, and selects a ball at random from it, what is the probability that she selects a red ball?

   (c) It turns out that she did select a red ball in part b) above. Given that information, what is the probability that she made her selection from bag 1?

Solutions:

(a)

$$\frac{3}{5} \times \frac{2}{3} \times \frac{1}{4}$$

(b) Use law of total probability:

$$\frac{1}{3} \times \frac{3}{5} + \frac{1}{3} \times \frac{2}{3} + \frac{1}{3} \times \frac{1}{4}$$

(c) Use Bayes:

$$P(1|R) = \frac{P(R|1)P(1)}{P(R|1)P(1) + P(R|2)P(2) + P(R|3)P(3)}.$$

Denominator is answer from part (b), numerator is

$$\frac{1}{3} \times \frac{3}{5}$$

3. There are three hiking trials that lead up to the summit of Hoosier hill, Indiana's highest point. On a typical Saturday afternoon in summer, hikers arrive at the summit from the (easy) Colfax trail at a rate of one every 12 minutes, from the (moderate) LaSalle trail on average once an hour, and from the (strenuous) Marquette trail on average once every 90 minutes. Hikers in Indiana are rugged individualists, and trek independently of each other. One Saturday at noon I take myself up to the summit to watch the hikers arrive.

   (a) Using an appropriate random variable to model the situation, calculate the probability that over the course of an hour I see no more than 8 hikers arrive from the Colfax trail. (You may leave your answer in the form of a sum.)

   (b) What is the probability that over the course of two hours, I see no more than 8 hikers reaching the summit in total (from all three trails)? (You may leave your answer in the form of a sum.)

   (c) What is the probability that in exactly three of the five hours that I am at the summit (noon to 1pm, 1 to 2pm, etc.), I see at least one hiker arriving at the summit from the Marquette trail? (You need not fully simplify your answer, but for full credit it should not contain a summation.)

Solutions:

(a) Use unit of time one hour. Average of 5 per hour, so use Poisson with parameter 5 to model X, number of arrivals.

$$P(X \leq 8) = \sum_{k=0}^{8} \frac{5^k}{k!} e^{-5}$$

(b) Use unit of time two hours. Average of $10 + 2 + (4/3) = 40/3$ hikers per two hours from all trails, so use Poisson with parameter $40/3$ to model Y, number of arrivals.

$$P(Y \leq 8) = \sum_{k=0}^{8} \frac{(40/3)^k}{k!} e^{-(40/3)}$$

(c) Model number of arrivals from Marquette as Poisson with parameter $2/3$. In single hour, probability of no arrivals is $e^{-2/3}$, so probability of at least one is $1 - e^{-2/3}$. Use Binomial with $n = 5$, $p = 1 - e^{-2/3}$ to get final probability:

$$\binom{5}{3}(1 - e^{-2/3})^3(e^{-2/3})^2$$

4. Let X be an exponential random variable with parameter λ. Let $Y = \sqrt{X}$.

   (a) Compute the distribution function of Y.

   (b) Compute the density function of Y.

   Solutions:

   (a) Range for X is 0 to infinity, so range for Y is 0 to infinity. For y in that range

$$P(Y \le y) = P(X \le y^2) = \int_0^{y^2} \lambda e^{-\lambda x}\, dx = [-e^{-\lambda x}]_0^{y^2} = 1 - e^{-\lambda y^2}$$

   Hence distribution of Y is $1 - e^{-\lambda y^2}$ if $y > 0$, 0 otherwise

   (b) Differentiate distribution. Density is 0 if $y \le 0$, and $2y\lambda e^{-\lambda y^2}$ if $y > 0$

5. A new life form, ET2, has been discovered in a remote part of the galaxy. The lifetime of a randomly chosen ET2 is an exponentially distributed random variable X with parameter λ.

   (a) Compute the cumulative distribution function F of X.

   (b) The distribution function satisfies the equation $F(1000) = 1/2$. (We say that 1000 is the *median* of the distribution.) Find λ.

   (c) How does $E(X)$ compare to the median, 1000?

   (d) Let $a > 0$. Find $b > a$ such that

$$P(X > b | X > a) = \frac{1}{2}.$$

   Solutions:

   (a) $F(x) = 1 - e^{-\lambda x}$ if $x > 0$, and 0 otherwise (direct from definition of exponential)

   (b) Have $1 - e^{-1000\lambda} = 1/2$, so $e^{1000\lambda} = 2$, so

$$\lambda = \frac{\log_e 2}{1000}$$

   (c) $E(X) = 1/\lambda = 1000/\log_e 2$ (using (b)), which is somewhat bigger than 1000.

   (d) Exponential is memoryless, so $P(X > b|X > a) = P(X > b - a)$. So $b - a = 1000$ (given in part (b)), and

$$b = a + 1000$$

6. A joint probability density function of random variables X and Y is given by the formula

$$f(x,y) = \begin{cases} \frac{2}{3}(2x+y) & \text{if } 0 \le x \le 1 \text{ and } 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find the marginal density $f_Y(y)$.

(b) Write down (but don't evaluate!) an integral whose value equals $P(3Y > X + 1)$.

(c) Write down (but don't evaluate!) an integral whose value equals $E((X-Y)^2 \ln Y)$.

Solutions:

(a) In general $f_Y(y) = \int_{x=-\infty}^{\infty} f(x,y)\,dx$. Here $f_Y(y) = 0$ if $y < 0$ or $y > 1$, and if $0 \le y \le 1$ then

$$f_Y(y) = \int_0^1 \frac{2}{3}(2x+y)\,dx = \frac{2}{3}[x^2 + xy]_{x=0}^1 = \frac{2}{3}(1+y)$$

(b) Draw line $3y = x + 1$ inside $[0,1]times[0,1]$ square (that's where the joint density is non-zero). It divides square into top and bottom parts. Top part is the one on which $3y > x + 1$ (use e.g. $(1,1)$ as a test point). Do double integral of joint density over that part. Easiest to slice piece into vertical strips (that way only one double integral needed):

$$P(3Y > X+1) = \int_{x=0}^1 \int_{y=\frac{x+1}{1}}^1 \frac{2}{3}(2x+y)\,dydx$$

(c) Use law of unconscious statistician, directly:

$$E((X-Y)^2 \ln Y) = \int_{x=0}^1 \int_{y=0}^1 (x-y)^2 \ln y \frac{2}{3}(2x+y)\,dydx$$

7. In Flatland, a three-sided die (marked with the numbers 1, 2, 3, one on each side) is a very popular toy. Two such dice are rolled. Let X be the number rolled on the first dice, and Y the maximum of the two numbers rolled.

(a) Find the joint probability mass function of X and Y.

$X \backslash Y$	**1**	**2**	**3**
1			
2			
3			

(b) Find $E(Y)$.

(c) Find the mass function of the random variable XY.

(d) Find $\text{Cov}(X,Y)$.

Solutions:

(a) Nine possibilities for the two rolls: $11, 12, 13, 21, 22, 23, 31, 32, 33$. Here's what each one leads to for the pair (X, Y):

$$(1,1),\ (1,2),\ (1,3),\ (2,2),\ (2,2),\ (2,3),\ (3,3),\ (3,3),\ (3,3)$$

Leads to joint mass:

$X \setminus Y$	**1**	**2**	**3**
1	1/9	1/9	1/9
2	0	2/9	1/9
3	0	0	3/9

(b) $Y = 1$ with probability 1/9, $Y = 2$ with probability 3/9, $Y = 3$ with probability 5/9

$$E(Y) = \frac{22}{9}$$

(c) Looking at the 6 non-zero entries of the joint mass, XY is

- 1 with probability 1/9
- 2 with probability 1/9
- 3 with probability 1/9
- 4 with probability 2/9
- 6 with probability 1/9
- 9 with probability 3/9.

This fully describes the mass function

(d) Know $E(Y) = 22/9$. Have

$$E(X) = 1\frac{1}{3} + 2\frac{1}{3} + 3\frac{1}{3} = 2$$

(X uniform on $\{1, , 2, 3\}$). Also have

$$E(XY) = 1\frac{1}{9} + 2\frac{1}{9} + 3\frac{1}{9} + 4\frac{2}{3} + 6\frac{1}{9} + 9\frac{3}{9} = \frac{47}{9}$$

So

$$\text{Cov}(X, Y) = E(XY) - E(X)E(Y) = \frac{47}{9} - 2\frac{22}{9} = \frac{3}{9}$$

8. Martin's supermarket sells grapefruits in plastic bags (four to a bag). The weight of a single grapefruit is a normal random variable with mean 0.5 lb and standard deviation 0.15 lb. The weight that can be sustained by a bag is also a normal random variable, with mean 2.2 lb and standard deviation 0.4 lb.

(a) What is the probability that a single grapefruit weighs more than .55 lbs?

(b) What is the probability that four grapefruits weight in total more than 2.2 lbs?

(c) What is the probability that a bag of four grapefruits will break when you try to lift it?

Solutions:

(a) $P(N(0.5, (0.15)^2) > .55) = P(Z > 1/3) = 0.3694...$

(b) Sum of independent normals is normal with mean equals sum of means, variance equals sum of variances, so weight of four independent grapefruit is $N(2, 4(0.15)^2) = N(2, (0.3)^2)$. (Notice: add variances, not standard deviations!). So want

$$P(N(2, (0.3)^2) > 2.2) = P(Z > 2/3) = 0.2525...$$

(c) Weight of grapefruits is $N(2, (0.3)^2)$. Capacity of bag is $N(2.2, (0.4)^2)$. Want

$$\begin{aligned} P(N(2, (0.3)^2) > N(2.2, (0.4)^2)) &= P(N(2, (0.3)^2) - N(2.2, (0.4)^2) > 0) \\ &= P(N(-0.2, (0.3)^2 + (0.4)^2) > 0) \\ &= P(N(-0.2, (0.5)^2 > 0) \\ &= P(Z > 0.4) \\ &= 0.3446... \end{aligned}$$

Notice: when subtracting independent normals, means subtract but variances add!

9. Let N be a positive integer, and let X be a discrete random variable with probability mass function given by

$$p(k) = \begin{cases} \frac{1}{N} & \text{if } k \in \{1, \ldots, N\} \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find the moment generating function of X. (You may leave it in summation form).

(b) Use the moment generating function to find $E(X)$. (You may leave your answer in summation form, but there's a bonus point for writing it in closed form).

(c) Let $X_1, \ldots, X_n$ be independent random variables, each having the same mass function p (as given above). What is the moment generating function of $X_1 + \ldots + X_n$?

Solutions:

(a) By definition $M_X(t) = E(e^{tX})$. By law of un/subconscious statistician,

$$E(e^{tX}) = \sum_{k=1}^{N} e^{tk} \frac{1}{N}$$

(b) Have $E(X) = M'_X(0)$ (derivative of moment generating function evaluated at 0).

$$M'_X(t) = \sum_{k=1}^{N} k e^{tk} \frac{1}{N}$$

so

$$E(X) = M_X'(0) = \sum_{k=1}^{N} k e^{t0} \frac{1}{N} = \sum_{k=1}^{N} k \frac{1}{N}$$

since $1 + 2 + \cdots + N = N(N+1)/2$ get

$$E(X) = \frac{N+1}{2}$$

(c) In general: when adding independent random variables, the moment generating functions multiply. So here, multiply answer from last part n times:

$$M_{X_1+\cdots+X_n}(t) = \left(\sum_{k=1}^{N} e^{tk} \frac{1}{N} \right)^n$$

10. When I make a phone call to one of my friends in Ireland, the length of the call (measured in minutes) is a random variable X with density function

$$f(x) = \begin{cases} \frac{3}{x^4} & \text{if } x \geq 1 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Compute $E(X)$, $E(X^2)$ and $\mathrm{Var}(X)$.

(b) I make 25 calls in a row. Assuming that call lengths are independent, use the central limit theorem to estimate the probability that I am on the phone for more than 42 minutes.

Solutions:

(a)

$$E(X) = \int_1^\infty x \frac{3}{x^4}\, dx = \int_1^\infty \frac{3}{x^3}\, dx = \frac{3}{2}$$
$$E(X^2) = \int_1^\infty x^2 \frac{3}{x^4}\, dx \int_1^\infty \frac{3}{x^2}\, dx = 3$$
$$\mathrm{Var}(X) = E(X^2) - E(X)^2 = \frac{3}{4}.$$

(b) Central limit theorem says that sum of $X_1 + \cdots + X_{25}$ (where X_i is length of ith call) is approximately normal with mean $25 \times (3/2) = 75/2$ and variance $25 \times (3/4) = 75/4$. So required probability is approximately

$$P(N(75/2, 75/4) > 42) \approx P(Z > 1.039) \approx 0.1494$$