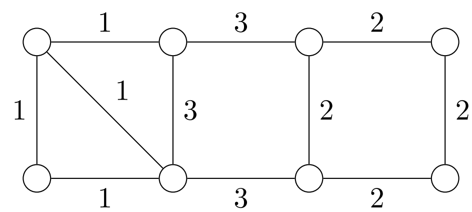
Math 40210: Basic Combinatorics, Fall 2012

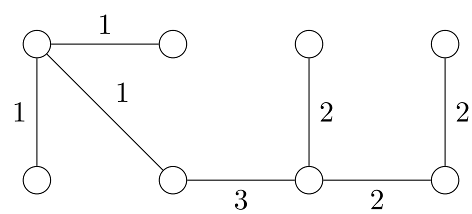
Midterm Exam, Monday, October 8

Solutions

1. (a) (5 pts.) Apply Kruskal's algorithm to find a minimal weight spanning tree in the following graph, and compute the weight of the tree you find.



Solution: One possibility is:



The weight of this tree is 12.

(b) (5 pts.) A connected graph has weights on its edges. Edge e has weight 1, and all other edges have weight greater than 1. Explain why the edge e must be in *every* minimum weight spanning tree of G. (For this question, you cannot just quote results we have proved about Kruskal's algorithm.)

Solution: Suppose T is a minimum weight spanning tree that does not include e. Then $T + e$ has a cycle that includes e. Let f be any other edge in the cycle. Now $T + e - f$ is a spanning tree; and because the weight of f is greater than that of e, the weight of $T + e - f$ is less than that of T, a contradiction (since T is minimum weight). So all minimum weight spanning trees include e.

2. (a) (5 pts.) A connected *bipartite* planar graph has $n \geq 3$ vertices. Show that it has at most $2n - 4$ edges.

Solution: If T is a tree, it has at most (actually exactly) $n - 1$ edges. For non-trees, so graphs with at least one region in their planar representations, we consider $\sum_R b(R)$, where R is a region of the planar representation, and $b(R)$ is the number of bounding edges. Since each edge bounds at most 2 regions, this sum is at most $2q$, where q is the number of edges. On the other hand, each region needs at least 4 bounding edges, since a bipartite G has no triangles. So the sum is at least $4r$, where r is the number of regions in the representation. Using Euler's formula $n - q + r = 2$, we get $r = 2 - n + q$, and so our two inequalities combine to give

$$4(2 - n + q) \leq 2q.$$

Rearranging terms we get $q \leq 2n - 4$. For $n \geq 3$, $n - 1 \leq 2n - 4$, so in all cases we get $q \leq 2n - 4$.

(b) (2 pts.) Give an example of a connected bipartite planar graph with n vertices and *exactly* $2n - 4$ edges. (A single n will do; think small!)

Solution: This bound is achieved by C_4 ($n = 4, q = 4$), or indeed by any graph of the form $K_{2,n-2}$.

(c) (3 pts.) A graph on vertex set $\{x_1, x_2, x_3, x_4, x_5\} \cup \{y_1, y_2, y_3, y_4, y_5\}$ has an edge from x_i to y_j if and only if $i \neq j$. Does this graph have a planar representation? Briefly justify your answer.

Solution: Such a graph would be bipartite and have $(4 \times 10)/2 = 20$ edges and 10 vertices. So $q = 20 > 16 = 2n - 4$, violating $q \leq 2n - 4$. So, no, such a graph is not planar.

3. (a) (4 pts.) Provide definitions for the following terms. Be sure to say what set of objects the term being defined applies to. To give you an idea of what I'm expecting, I've done the first one.

   i. **Size**: The size <u>*of a graph*</u> is the number of edges in the graph.

   ii. **Eccentricity**: **Solution**: The eccentricity of a vertex is the distance from the vertex to the furthest away vertex in the graph.

   iii. **Diameter**: **Solution**: The diameter of a graph is the value of the largest eccentricity.

   (b) (6 pts.) Show that if the diameter of $\overline{G}$ is at least 3, then G is connected.

   Solution: Suppose G is not connected. Then the vertex set can be partitioned into $X \cup Y$ with no edges going from X to Y. But then in $\overline{G}$, there is an edge from every $x \in X$ to every $y \in Y$. This means that every vertex has eccentricity at most 2 in $\overline{G}$ (for a vertex in X, it takes one step to get to any vertex in Y and one more to get to any vertex in X; the same for any vertex in Y). So the diameter of $\overline{G}$ is at most 2, a contradiction.

4. (a) (4 pts.) Give the definitions of a *Hamilton path* and a *Hamilton cycle* in a graph.

   - **Hamilton path**:
     Solution: A Hamilton path is a path that visits every vertex of the graph.
   - **Hamilton cycle**:
     Solution: A Hamilton cycle is a cycle that visits every vertex of the graph.

(b) (3 pts.) Use Dirac's theorem to show that an r-regular graph with 16 vertices and 96 edges has a Hamilton cycle. (**Hint**: first figure out what r must be.)

Solution: Dirac says that for G with $n \geq 3$ and $\delta \geq n/2$ (δ the minimum degree), there is a Hamilton cycle. Here $n = 16$. Since the sum of the degrees is twice the number of edges, and all degrees are the same, we have $16d = 2 \times 96$ and so $d = 12$ (where d is the degree of every vertex). So $\delta = 12 > 8$, and there is a Hamilton cycle.

(c) (3 pts.) Use Dirac's theorem to show that the graph from the last part in fact has 3 Hamilton cycles that use completely disjoint sets of edges.

Solution: We want to find three edge-disjoint Hamilton cycles. So, remove the first one we found. This drops every degree by exactly 2, leaving us with a 16-vertex graph with minimum degree 10. Dirac still applies, so there is a Hamilton cycle in the new graph (which is clearly disjoint from the first). Remove this, to get a 16-vertex graph with minimum degree 8. Dirac *still* applies, so there is a Hamilton cycle in the new graph (which is clearly disjoint from the first two). We can't go any further; after removing the third Hamilton cycle, the minimum degree is now 6 and Dirac no longer applies.

5. (a) (3 pts.) State Cayley's formula on the number of labeled trees on n vertices.

Solution: There are n^{n-2} labeled trees on n vertices.

(b) (7 pts.) How many labeled trees are there on the vertices $\{1, 2, \ldots, n\}$ that have vertex 1 as a leaf? (You may use any facts that you know about Cayley's formula and/or Prüfer codes, but please state them clearly!)

Solution: There are two possible solutions. First, there are $(n-1)^{(n-1)-3} = (n-1)^{n-3}$ labeled trees on $\{2, \ldots, n\}$. To each of these we can add 1 as a leaf in $n-1$ ways (by joining it to any of $2, \ldots, n$). This gives $(n-1)(n-1)^{n-3} = (n-1)^{n-2}$ trees with 1 as a leaf.

Second, 1 being a leaf means that 1 doesn't appear in the Prüfer code (the number of times a vertex appears in the code is its degree minus 1). So the construct a Prüfer code of a tree with 1 as a leaf, we just choose each of the $n-2$ entries of the code from the set $\{2, \ldots, n\}$. There are $(n-1)^{n-2}$ such choices.