Basic Combinatorics

Math 40210, Section 01 — Fall 2012

Homework 3 — Solutions

- **1.3.3 1**: We prove this by induction on the number of edges. Since G is connected, it has at least $n-1$ edges, so this is our base case. The base case is trivial: G is a tree so it is itself a spanning tree. For connected G with more than $n-1$ edges, there must be a cycle (if it had no cycles, it would be a tree, and so have only $n-1$ edges). Remove one edge from a cycle; the resulting graph G' must be connected, and has fewer edges, so by induction it has a spanning tree. This spanning tree is also a spanning tree of the original graph G.

- **1.3.3 2**: Suppose that G is a tree. Then we know (from definition, and a theorem in class) that it is connected and has $n-1$ edges. Let T and T' be two spanning trees of G. We want to show that $T = T'$ (i.e., that T only has one spanning tree). Suppose not. Then there must be an edge e that is in one of T, T' but not in the other; assume without loss of generality that $e \in E(T)$. But then the $n-1$ edges of T', together with e, give at least n edges in T, a contradiction. So G has a unique spanning tree.

  Conversely, suppose that G is connected and has a unique spanning tree, call it T. We claim that $G = T$ (so G is in fact a tree). If not, let e be an edge of G that is not in T. Adding e to T creates a cycle; removing any edge (other than e) from that cycle gives a new spanning tree (different from T), a contradiction. So G is a tree.

- **1.3.3 4**: Suppose that e is a bridge, and that on its removal the graph breaks into two components, one with vertices V_1, the other with vertices V_2. Let T be a spanning tree of G. If T does not include e, then there is no path in T from any $v \in V_1$ to and $u \in V_2$ (since all such paths must use e), contradicting the fact that T is a tree. So $e \in E(T)$.

  Conversely, suppose that e is in every spanning tree of G. If $G - e$ is connected, then it has a spanning tree, which does not include e and is also clearly a spanning tree of G, a contradiction. So $G - e$ is not connected, that is, e is a bridge.

- **1.3.3 5**: The graph on the right has a spanning tree, obtained via Kruskal, using edges with weights 1, 2, 4, 5, 6, 7, 8 and 10, giving a total weight 43. This is the unique minimum weight spanning tree. To see that it is unique: the smallest weight of a set of eight edges (the number in a spanning tree for a graph on 9 vertices) that fails to use the edge weighted 1 is $2+3+4+5+6+7+8+9 > 43$. So the edge weighted 1 must be in any minimum weight spanning tree. The smallest weight of a set of eight edges uses the edge weighted 1 but that fails to use the edge weighted 2 is $1+3+4+5+6+7+8+9=43$; but this set of edges (the unique such) isn't a tree. So the edge weighted 2 must also be in any minimum weight spanning tree, and since it forms a cycle with the edges weighted 1 and 2, the edge weighted 3 cannot be in a minimum weight spanning tree. Now that we have decided that the edges weighted 1 and 2 must appear in a minimum weight spanning tree, note that one of the edges weighted 4, 5, 8 or 9 must not appear (else we would have a cycle). If we choose to leave out the edge weighted 8, then the smallest sum of eight edge weights is $1+2+4+5+6+7+9+10>43$; and the same thing happens if we chose to leave out the edge weighted 5, or 4. If we chose to leave out the edge weighted 9, then the smallest sum of eight edge weights is $1+2+4+5+6+7+8+10=43$; this is the unique such sum, and it is a tree (the spanning tree given by Kruskal), so we've proven that the tree has a unique minimum weight spanning tree.

For the graph on the left, Kruskal gives a minimum weight spanning tree consisting of the edges with weights 2, 3 and 5 (both in each case), giving a total weight of 20. This is also unique. To see this: note that we must lose at least one edge from the triangle in the northeast (with edges weighted 2, 3 and 4), and we must lose at least one edge from the triangle in the northeast (with edges weighted 2, 3 and 4). So the minimum possible weight of a spanning tree is the sum of the eight lightest edges, minus the heaviest edges from each of these cycles; that's $2+2+3+3+4+4+5+5-4-4=20$; and this sum can clearly be achieved uniquely (by the tree given by Kruskal).

- **1.3.4 1**: The second statement implies the first, since leaves have degree 1. We prove the statement by induction on n, the number of vertices of T.

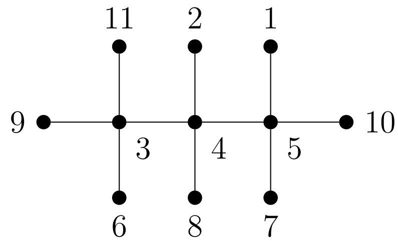
  For $n=2$, each vertex is a leaf, and both appear 0 times in the Prüfer code, so the base case is ok.

  For $n>2$, let i be the label of the lowest leaf, and j the label of its unique neighbour. Because i is immediately deleted, it never appears in the Prüfer code. The unique neighbour of i, that is j, appears once at the beginning of the Prüfer code. After that, the rest of the Prüfer is exactly the Prüfer code of $T'=T-i$ (on label set $\{1,2,\ldots,n\}\setminus\{j\}$). By induction, in that rest of the Prüfer code, each label k appears exactly $d_{T'}(k)-1$ times. For $k\neq i,j$, we have $d_T(k)-1=d_{T'}(k)-1$, so all of these labels appear $d_T(k)-1$ times. Since in T' one neighbour of j has been lost, we have $d_{T'}(k)-1=d_T(k)-2$; that's the number of times label k appears in the last part of the Prüfer code. But it also appears once at the very beginning, for a total of $d_T(k)-1$ appearances. Finally, as already observed, label i appears zero (which is its degree minus one) times in the Prüfer code. That accounts for all the vertices, and the induction is complete.

- **1.3.4 2**: The left: $(2,2,1,3,1,4,4,1,5)$. The right: $(5,8,4,3,3,3,9)$.

- **1.3.4 3**:

- **1.3.4 4**: If a tree on n vertices $1,\ldots,n$ has Prüfer sequence $(c,c,c,\ldots,c)$ then c must have degree $(n-2)+1=n-1$ and everything else must be a leaf. So the tree is a star, with center vertex c.



- **1.3.4 5**: If a tree on n vertices $1, \ldots, n$ has Prüfer sequence in which all entries are distinct, then there must be exactly two entries missing, each of which must be leaves, and all the other entries must correspond to vertices of degree 2; so the tree is a path.

- **1.3.4 6**: We want to count the number of trees that *miss* a particular edge of K_n; it's slightly easier to calculate t_e, the number of trees that *use* a particular edge e. We'll use the fact that t_e doesn't actually depend on e (i.e., for the total count it doesn't matter which edge you choose to miss/use). This allows us to use a double-count, similar to the proof that the sum of the degrees in a graph equals twice the number of edges. On the one hand, we have

$$\sum_{e \in E(K_n)} t_e = \frac{n(n-1)}{2} t$$

where t is the common value taken on by all the t_e. On the other hand,

$$\begin{aligned}
\sum_{e \in E(K_n)} t_e &= \sum_{e \in E(K_n)} \sum_{T} \mathbf{1}_{\{e \in E(T)\}} \\
&= \sum_{T} \sum_{e \in E(K_n)} \mathbf{1}_{\{e \in E(T)\}} \\
&= \sum_{T} (n-1) \\
&= (n-1)n^{n-2}
\end{aligned}$$

where $\sum_T$ indicates sum over all spanning trees of K_n. In the second equality, we reversed order of summation; in the third we used the fact that all trees on n vertices have $n-1$ edges; and in the last we used Cayley's formula.

Combining the first displayed equation with the second we get

$$t = 2n^{n-3}$$

and so the number of trees that *miss* a particular edge is $n^{n-2} - t = (n-2)n^{n-3}$.

Here's a quicker description: K_n has n^{n-2} spanning trees, each with $n-1$ edges, so counting the total number of edges over all spanning trees, we get $(n-1)n^{n-2}$. Each edge is counted the same number of times in this grand sum (by symmetry), and K_n has $(n^2-n)/2$ edges, so each edge is counted $(n-1)n^{n-2}/((n^2-n)/2) = 2n^{n-3}$ times. So this is the number of times that a particular edge appears in a spanning tree; the number of times it does not appear is $n^{n-2} - 2n^{n-3} = (n-2)n^{n-3}$. (And why does K_n

have $(n^2 - n)/2$ edges? The sum of the vertex degrees is $n(n-1) = n^2 - n$, and this is twice the number of edges.)

- **1.3.4 7**: Here's what I think is the most natural "follow your nose" solution:

  Cayley's formula counts the number of spanning trees in K_n. The Laplacian matrix L of K_n is the n by n matrix with $n-1$'s down the diagonal and -1's everywhere else. The 1-1 cofactor of this is the determinant of the $n-1$ by $n-1$ matrix with $n-1$'s down the diagonal and -1's everywhere else. We prove by induction that this is n^{n-2}. The base case, say $n = 2$, is trivial. For larger n: we use row-reduction to help us compute the determinant. Start by using the first row of the matrix to clear all the off-diagonal entries of the first column (turn them to zero). Away from the first row and column, the entries that had been $n-1$ become $(n-1) - 1/(n-1)$, and the entries that had been -1 become $-1-1/(n-1)$. The determinant remains unchanged (adding a multiple of one row to another doesn't alter the determinant). Expanding now along the first column, we get that the determinant is $(n-1)$ times the determinant of the $n-2$ by $n-2$ matrix with $(n-1)-1/(n-1)$'s down the diagonal and $-1-1/(n-1)$'s everywhere else. Multiply each row by $(n-1)/n$; after a little algebra, you see that the matrix becomes the $n-2$ by $n-2$ matrix with $n-2$'s down the diagonal and -1's everywhere else. By induction, the determinant of this matrix is $(n-1)^{n-3}$. But multiplying a row by a constant multiplies the determinant by the same constant, so to get back to the determinant of the $n-2$ by $n-2$ matrix with $(n-1)-1/(n-1)$'s down the diagonal and $-1-1/(n-1)$'s everywhere else, we have to *divide* $(n-1)^{n-3}$ by $((n-1)/n)^{n-2}$ (one $(n-1)/n$ for each row). Putting things together, we get that the determinant of the original matrix ($n-1$ by $n-1$ with $n-1$'s down the diagonal and -1's everywhere else) is

  $$(n-1) \times \frac{(n-1)^{n-3}}{\left(\frac{n-1}{n}\right)^{n-2}} = n^{n-2},$$

  completing the induction.

  Nicholas Wawrykow pointed out an alternate approach, along the same lines, that involves much less algebra. Begin by deleting first row and *last* column. We need to calculate $(-1)^{n+1}$ times the determinant of this minor. Use the *last row* of the minor to eliminate (turn to 0) all the entries in the first column of the minor; this doesn't change the determinant. The resulting matrix has zeroes down the first column, except for the last entry; -1's along the last row, and the rest is n times the $(n-2)$ by $(n-2)$ identity matrix. Expanding along the first column, we thus get that the determinant is $(-1)(-1)^{1+n-1}n^{n-2}$; so the quantity being computed by the Matrix Tree Theorem is n^{n-2} (all the -1's cancel).