Basic Combinatorics

Math 40210, Section 01 — Fall 2012

Homework 6 — Solutions

- **1.7.1 1**: It does not have a perfect matching. A perfect matching is one which saturates all vertices, and so in particular must saturate the vertex at the center. Suppose, wlog, that this vertex is saturated by the edge dropping down to the bottom 5 vertices. The matching then cannot use either of the two edges going up to the upper right or upper left groups of 5 vertices. So for each of these groups of 5, vertices can only get saturated using edges within the group. This is not going to be possible for either group, since 5 is odd. Once two edges have been chosen, the 5th vertex cannot get saturated without one of the other 4 vertices getting re-used.

- **1.7.1 2a**: There is no maximal matching in C_{10} of size 3. Each edge in a matching removes three edges from consideration (the edge in the matching, and its right and left neighbors), so 3 edges can remove at most 9 edges from consideration, leaving at least one that can be added. But there *is* a maximal matching of size 4, as shown in figure 1 of the figures page.

- **1.7.1 2b**: There is no maximal matching in C_{11} of size 3, for the same reason as described above; but there *is* a maximal matching of size 4, as shown in figure 2 of the figures page.

- **1.7.1 2c**: If n is of the form $3k$, there is a maximal matching of size k (take every third edge), but none smaller (for the reason described in part a)). If n is of the form $3k+1$, there is a maximal matching of size $k+1$ (take edges v_1v_2, v_4v_5, v_7v_8, etc., until k edges have been taken, the last being $v_{3k-2}v_{3k-1}$; then add $v_{3k}v_{3k+1}$ as the $(k+1)$st edge), but none smaller (for the reason described in part a)). If n is of the form $3k+2$, there is a maximal matching of size $k+1$ (take edges v_1v_2, v_4v_5, v_7v_8, etc., until k edges have been taken, the last being $v_{3k-2}v_{3k-1}$; then add either $v_{3k}v_{3k+1}$ or $v_{3k+1}v_{3k+2}$ as the $(k+1)$st edge), but none smaller (for the reason described in part a)).

- **1.7.2 1**: For each of the two graphs, there are lots of M-alternating paths that are not M-augmenting; a silly way to find one is to just pick a single edge of M ... it's a path (of length 1) but is clearly not M-augmenting. There are *no* M-augmenting paths in the first graph (bottom left) - every edge not in the matching has both endvertices M-saturated, so an M-augmenting path cannot begin. There *are* M-augmenting paths in the second graph (bottom right) - one of them is shown in figure 3 of the figures page.

- **1.7.2 2a**: The condition is met. An example of an SDR is this: take i from the ith set listed, $i = 1, \ldots, 5$.
- **1.7.2 2b**: The condition is met. An example of an SDR is this: take 4 from $\{1, 2, 4\}$, then 2 from $\{2, 4\}$, then 3 from $\{2, 3\}$, and finally 1 from $\{1, 2, 3\}$.
- **1.7.2 2c**: The condition is violated by the sets $\{1, 2\}$, $\{2, 3\}$, $\{1, 2, 3\}$ and $\{1, 3\}$ - four sets which between them have a union of size 3. So there is no SDR.
- **1.7.2 4**: The statement is false. For example, if $S_1 = \{1, 2\}$, $S_2 = \{1, 3\}$,, $S_3 = \{1, 4\}$,, $S_4 = \{2, 3\}$, $S_5 = \{2, 4\}$ and $S_6 = \{3, 4\}$ (so $n = 4$, $r = 6$ and the common cardinality is 2), then there is clearly no SDR.
- **1.7.2 6**: We show that Hall's condition is satisfied. Let $S \subseteq X$ be given. Look at the subgraph of G that is induced by the vertices $S \cup N(S)$ (so from $N(S)$, the set of neighbors of S in Y, we only look at edges back to S). Let $e(S)$ be the number of edges in this graph. Going from S to $N(S)$, we have $e(S) \geq \delta_X|S|$ (each $x \in S$ contributes at least δ_X edges). Going from $N(S)$ to S, we have $e(S) \leq \Delta_Y|N(S)|$ (each $y \in N(S)$ contributes at most Δ_Y edges). Putting these two together we get $\delta_X|S| \leq \Delta_Y|N(S)| \leq \delta_X|N(S)|$ (the last inequality using $\delta_X \geq \Delta_Y$), and so $|S| \leq |N(S)|$. This is Hall's condition; so by Hall's theorem there is a matching of X into Y.
- **1.7.3, 1**: Let $G = X \cup Y$ be a bipartite graph that satisfies Hall's condition, $|N(S)| \geq |S|$ for all $S \subseteq X$. We want to show that there is a matching of X into Y. Let M be a maximum matching. By the König-Egerváry theorem, there is an edge cover of size $|M|$. Let A be the set of vertices of the edge cover that are in X, $B = X \setminus A$, and C the set of vertices of the edge cover that are in Y. We have a few inequalities and equalities:
  - $|A| + |B| = |X|$ (by definition of A and B)
  - $|B| \leq |N(B)| \leq |C|$ (the first of these is Hall's condition; the second is because $N(B) \subseteq C$. Indeed, if $x \in B$ has a neighbor $y \notin C$, then xy is an edge not covered by the edge cover).
  - $|A| + |C| = |M|$ (this is König-Egerváry).

  Combining the first and second above we get $|X| \leq |A| + |C|$; using the third, we get $|X| \leq |M|$. In other words, the maximum matching has size at least the size of X. But since each edge of M uses a vertex of X, and each vertex can be used at most once, it must be that $|X| = |M|$, and M saturates X. This proves Hall.
- **1.7.3, 2**: By the König-Egerváry theorem, the maximum matching in the subgraph equals the minimum edge cover size. So if we can show that no set of $k - 1$ vertices can form an edge cover of the subgraph, so the minimum edge cover has size at least k, we have shown that the maximum matching size is at least k and so there has to be a matching of size k. Since the maximum degree of $K_{n,n}$ is n, each vertex in the subgraph covers at most n edges, so $k - 1$ vertices can cover at most $(k - 1)n$ edges; but the subgraph has more than $(k - 1)n$ edges, so no set of $k - 1$ vertices can form an edge cover.

- **1.7.3, 3**: It seems sensible to try to translate the problem to one on a bipartite graph where zeros of the matrix correspond to edges of the graph, independent sets of zeros correspond to matchings, rows and columns correspond to vertices, and a covering of the zeros corresponds to an edge cover. So: form a bipartite graph $G = X \cup Y$ where X consists of the m rows of the matrix, Y is the n columns, and there is an edge from $i \in X$ to $j \in Y$ if the ij entry is zero. A matching in this graph is exactly a set of zeros no two of which are in the same row or column, and an edge cover is exactly a set of rows and columns that covers all zeros. So the statement of the König-Egerváry theorem (maximum matching size = minimum cover size) translates directly to maximum number of independent zeros = minimum size of a covering family of rows and columns.

- **1.7.4, 7**: Remember that a perfect matching is one that saturates all the vertices. So every leaf of the tree must be saturated, and so the edge leaving that leaf must be used in the matching, and therefore none of the edges leaving the neighbor of the leaf, except the one going back to the leaf, are allowed to be use. This suggests a proof by induction, on the number of edges. If T has no edges, or 1, then it is obvious that there is at most one (in fact exactly one) perfect matching. So assume that our tree has more than one edge. Start building a possible perfect matching by including all the edges leaving leafs (note that this may kill us: there may be two leaves adjacent to the same vertex. If so, we automatically have a tree with *no* perfect matchings). Then (if we have succeeded thus far) delete from T all those edges and all other edges leaving vertices that are joined to leaves, as well as the leaf vertices and their neighbors. The result graph is still acyclic, so all of its components are trees, and each tree has fewer edges than the initial tree. To extend the matching we've started to a perfect matching of the original tree, we need to find a perfect matching in each of the components, and by induction there is at most one way to this for each component, so there is at most one way to do it simultaneously for the collection of components. This finishes the induction step.