Basic Combinatorics

Math 40210, Section 01 — Fall 2012

Homework 8 — Solutions

- **1.8.1 1**: K_n has $\binom{n}{2}$ edges, each one of which can be given one of two colors; so K_n has $2^{\binom{n}{2}}$ 2-edge-colorings.

- **1.8.1 3**: Let $\chi : E(K_k) \to \{r, b\}$ be a 2-colouring of the edges of K_k with two colors, r and b. We want to show that *either* there is a complete subgraph on 2 vertices, all of whose edges are colored r, or a complete subgraph on k vertices, all of whose edges are colored b. If there is even a single edge coloured r, we have the first. If not, then all edges are colored blue, and we have the second. This shows that $R(2, k) \leq k$. To see that $R(2, k) = k$, we exhibit a 2-coloring of the edges of K_{k-1} that contains neither a complete subgraph on 2 vertices, all of whose edges are colored r, or a complete subgraph on k vertices, all of whose edges are colored b. This is easy: take the coloring in which every edge is coloured b.

- **1.8.1 5**: First the inequality. If $R(p, q) = N$, then every 2-colouring of the edges of K_N with two colors, r and b contains *either* a complete subgraph on p vertices, all of whose edges are colored r, or a complete subgraph on q vertices, all of whose edges are colored b. If the former, then it contains a complete subgraph on p' vertices, all of whose edges are colored r (just take any p' vertices from the totally-r-colored K_p), or a complete subgraph on q' vertices, all of whose edges are colored b. So $R(p', q') \leq N$.

  Next, the strictness. Suppose at least one of $p' = p$, $q' = q$ fails. Without loss of generality, $p' < p$. Suppose, for a contradiction, that $R(p', q') = R(p, q) = N$. This means that there exists a 2-colouring of the edges of K_{N-1} which contains *neither* a complete subgraph on p' vertices, all of whose edges are colored r, or a complete subgraph on q' vertices, all of whose edges are colored b. To this graph, add an Nth vertex, and join it all of the $N - 1$ existing vertices with an edge colored r. The resulting 2-coloring of the edges of K_N does not contain a complete subgraph on $p' + 1$ vertices, all of whose edges are colored r (if it did, the original coloring would have contained a complete subgraph on p' vertices, all of whose edges are colored r); so, since $p' < p$, the coloring just constructed does not contain a complete subgraph on p vertices, all of whose edges are colored r. Nor does it contain a complete subgraph on q' vertices, all of whose edges are colored b (if it did, the original coloring would have contained a complete subgraph on q' vertices, all of whose edges are colored b); so, since $q' \leq q$, the coloring just constructed does not contain a complete subgraph on q vertices, all of whose edges are colored b. It follows that $R(p, q) > N$, a contradiction.

- **1.8.2 5**: We proceed by induction on q. The base case is

$$R(3,3) \leq \frac{3^2+3}{2} = 6,$$

which is true (in fact, $R(3,3) = 6$).

For the induction step, fix $q > 3$. We have

$$\begin{aligned} R(3,q) &\leq R(2,q) + R(3,q-1) \quad \text{(Theorem 1.64)} \\ &\leq q + \frac{(q-1)^2+3}{2} \quad \text{(An easy fact, and induction)} \\ &\leq \frac{q^2+4}{2} \quad \text{(Combining).} \end{aligned}$$

This does not seem to do the job! *But*, part of Theorem 1.64 is that if both terms on the right-hand side are even, then the inequality is strict. We repeat the induction using this fact. If q (and so $R(2,q) = q$) is even, and *also* $R(3,q-1)$ is even, then we can put a -1 on the right-hand side of each inequality above, and we end up getting

$$R(3,q) \leq \frac{q^2+2}{2} \leq \frac{q^2+3}{2}.$$

If, on the other hand, q is even but $R(3,q-1)$ is odd, then note the following: $(q-1)^2$ leaves a remainder of 1 on division by 4 (to check this, look at $(2k-1)^2 = 4k^2 - 4k + 1$), so $(q-1)^2 + 3$ leaves a remainder of 0 on division by 4, so $((q-1)^2+3)/2$ is even. So in this case, we may, knowing that $R(3,q-1)$ is odd, replace the inductive upper bound by $\frac{(q-1)^2+3}{2} - 1$, which again leads us to a good bound in the end.

What is q is odd? Then our upper bound for $R(3,q-1)$, namely $\frac{(q-1)^2+3}{2}$, is not an integer (it's an integer plus $1/2$). So we may replace it by $\frac{(q-1)^2+3}{2} - 1/2$, leading to

$$R(3,q) \leq \frac{q^2+3}{2}.$$

- **2.4 3**: The question as asked is a little hard to follow. Here's a rephrasing: if you have n pigeonholes, labeled 1 through n, and pigeonhole i has "capacity" m_i, then if you have $M = m_1 + \ldots + m_n - n + 1$ pigeons distributed among the pigeonhole, there must be one pigeonhole that is filled to capacity (at least). In the usual pigeonhole principle each of the m_i's are 2, and M becomes $2n - n + 1 = n + 1$.

  The proof is easy: if, for each i, pigeonhole i gets fewer than m_i pigeons, then it gets at most $m_i - 1$, so the total number of pigeons accounted for is at most $(m_1 - 1) + \ldots + (m_n - 1) = m_1 + \ldots + m_n - n = M - 1$. So with M pigeons, we must have at least one i with the ith pigeonhole having at least m_i pigeons.

- **2.4 4**: The best pitcher has the most possible strikeouts when the other three have the minimum possible, that is 40; in this case the best pitcher has 177 strikeouts. The best pitcher has the least possible strikeouts when the worst pitcher has the least possible number, that is 49, and the remaining 248 are distributed as near equally as possible among the remaining three, that is 82, 83 and 83.

So the range of possible strikeouts for the best pitcher is between 83 and 177.

- **2.4 5**: The question must mean that we take the sum of difference of two *distinct* numbers; if we are allowed to take the difference or sum of a single number with itself, the answer is trivially $m = 1$, since any number x has $x - x = 0$, a multiple of 10.

  The 6 numbers 0,1,2,3,4,5 show that we need at least $m = 7$. To see that $m = 7$ makes the statement valid, create 6 pigeonholes, labeled by:

  1. ends with a 0 or (that is, has 0 as last digit in decimal representation)
  2. ends with a 1 or a 9
  3. ends with a 2 or an 8
  4. ends with a 3 or a 7
  5. ends with a 4 or a 6
  6. ends with a 5.

  If two numbers end up in the same pigeonhole, either their sum or their difference is a multiple of 10; so by the pigeonhole principle we can certainly take $m = 7$.

- **2.4 7**: Label the numbers, in the order they appear around the circle, $a_1, a_2, \ldots, a_n$ (so a_1 is beside a_n). If you look at the numbers in consecutive blocks of length k, going the whole way around the circle, then each number appears exactly k times. So if you sum up all the numbers that appear in these blocks (counting as different appearances of numbers in different blocks), you get a sum of exactly

  $$k(a_1 + \ldots + a_n) = k(1 + 2 + \ldots + n) = \frac{kn(n+1)}{2}.$$

  That means that the *average* sum of a consecutive block of k is exactly $k(n + 1)/2$. Since at least one number in a sequence must be as least as large as the average, there must be at least one block of length k whose sum is at least $k(n + 1)/2$. But since the sums must all be integers, this large sum must be at least the first integer that is at least as large as $k(n+1)/2$, that is $\lceil k(n + 1)/2\rceil$.

- **2.4 10**: Here's one possible arrangement: take all numbers between 1 and mn that leave a remainder of 1 on division by m, and list them in decreasing order; then take all numbers that leave a remainder of 2 on division by m, and list them in decreasing order; and so on, all the way to all numbers that leave a remainder of $m - 1$ on division by m, and finally all numbers that leave a remainder of 0 on division by m, always listing the block of numbers in decreasing order. Here's what this looks like for $n = 5$ and $m = 5$:

  $$\begin{array}{ccccc} 21 & 16 & 11 & 6 & 1 \\ 22 & 17 & 12 & 7 & 2 \\ 23 & 18 & 13 & 8 & 3 \\ 24 & 19 & 14 & 9 & 4 \\ 25 & 20 & 15 & 10 & 5. \end{array}$$

An increasing sequence can only take one number from each row, so there can be at most m terms in it; a decreasing sequence can only take one number from each column, so there can be at most n terms in it.

- **2.4 13**: This is a classic pigeonhole problem that is easy once you know the solution, but coming up with the solution is quite tough.

  Consider the n sums:

$$\begin{array}{l} a_1 \\ a_1 + a_2 \\ a_1 + a_2 + a_3 \\ \dots \\ a_1 + a_2 + \dots + a_n. \end{array}$$

  If one of these is a multiple of n, we are done. If not, these are n sums that are shared between $n-1$ remainders on division by n. By the pigeonhole principle, two of them, $a_1 + \dots a_\ell$ and $a_1 + \dots + a_k$ (with, wlog, $k > \ell$), have the same remainder on division by n. If follows that the difference between these two sums is a multiple of n, that is, $a_{\ell+1} + \dots + a_k$ is a multiple of n, and we are done.

- **2.5 1**: S is set of all flags, A_B those with blue background, A_S those with stripes, A_P those with plant or animal. Know:

$$|S| = 50$$

$$|A_B| = 30; |A_S| = 12; |A_P| = 26$$

$$|A_B \cap A_S| = 9; |A_B \cap A_P| = 23; |A_S \cap A_P| = 3$$

$$|A_B \cap A_S \cap A_P| = 2$$

  We want to find $|(A_B \cap A_P \cap A_S)^c|$. A direct application of Inclusion-exclusion yields:

$$\begin{aligned} |(A_B \cap A_P \cap A_S)^c| &= 50 - (30 + 12 + 26) + (9 + 23 + 3) - 2 \\ &= 15. \end{aligned}$$

- **2.5 6**: Let A_S, A_C, A_H and A_D denote, respectively, the number of five cards hands that **don't** include a spade, club, heart or diamond. The size of each of these sets is $\binom{39}{5}$ (for A_S, for example, we have to choose the five cards from among the 39 non-spades). The size of the intersection of any two of these sets is $\binom{26}{5}$ (for $A_S \cap A_C$, for example, we have to choose the five cards from among the 26 non-black-cards). The size of the intersection of any three of these sets is $\binom{13}{5}$, and the size of the intersection of all four is $\binom{0}{5} = 0$. Inclusion-exclusion gives that the number of five-card hands with at least one from each suit is

$$\begin{aligned} |(A_C \cap A_S \cap A_H \cap A_D)^c| &= \binom{52}{5} - 4\binom{39}{5} + 6\binom{26}{5} - 4\binom{13}{5} + 0 \\ &= 685464. \end{aligned}$$

  This is about 26% or all five-card hands.

- **2.5 7**: In general when we expand out a product of the form

$$\prod_{i=1}^{r}(a_i + b_i)$$

we get a sum of the form

$$\sum_{S\subseteq\{1,\dots,r\}} \prod_{i\in S} b_i \prod_{i\notin S} a_i.$$

This is because to get a term in the expanded sum, we have to make a choice for each of the r terms in the product: do we take the b_i or the a_i? We can encode the choices we made for a particular term by a subset S of $\{1,\dots,r\}$ corresponding to where we made the choices of b_i's; each of the 2^r subsets leads to a different term of the expanded sum.

In this case, each a_i equals 1, and each b_i equals $-\alpha_i$. The term corresponding to the empty set is exactly 1; the term corresponding to a set $S = \{i\}$ is exactly $-\alpha_i$ and we have one for each i; the term corresponding to a set $S = \{i,j\}$ with $i < j$ is exactly $-\alpha_i\alpha_j$ and we have one for each $i < j$; the term corresponding to a set $S = \{i,j,k\}$ with $i < j < k$ is exactly $-\alpha_i\alpha_j\alpha_k$ and we have one for each $i < j < k$, and so on up to the term corresponding to $S = \{1,\dots,r\}$, which is $(-1)^r\alpha_1\dots\alpha_r$.

- **2.5 8a**: If m and n are relatively prime we have

$$m = p_1^{a_1}\dots p_r^{a_r}$$

and

$$n = q_1^{b_1}\dots q_s^{b_s}$$

with no primes in common among the p's and q's. So the prime factorization of mn is

$$mn = p_1^{a_1}\dots p_r^{a_r}q_1^{b_1}\dots q_s^{b_s}.$$

By our formula for the ϕ function we have

$$\begin{aligned}
\phi(mn) &= mn\left(1-\frac{1}{p_1}\right)\dots\left(1-\frac{1}{p_r}\right)\left(1-\frac{1}{q_1}\right)\dots\left(1-\frac{1}{q_s}\right)\\
&= m\left(1-\frac{1}{p_1}\right)\dots\left(1-\frac{1}{p_r}\right)n\left(1-\frac{1}{q_1}\right)\dots\left(1-\frac{1}{q_s}\right)\\
&= \phi(m)\phi(n).
\end{aligned}$$

- **2.5 8b**: If m and n are not relatively prime, then in the notation of the last solution there is at least one coincidence between the p' and the q's. We still have

$$\phi(m)\phi(n) = mn\left(1-\frac{1}{p_1}\right)\dots\left(1-\frac{1}{p_r}\right)\left(1-\frac{1}{q_1}\right)\dots\left(1-\frac{1}{q_s}\right),$$

but $\phi(mn)$ is missing at least one of the terms or the form $(1-1/p_i)$ (if $p_1 = q_1$, for example, we would include $(1-1/q_1)$ in calculating $\phi(mn)$, but not $(1-1/p_1)$, since this would lead to an unwanted duplication). This shows that the two sides are not equal. Since $\phi(mn)$ is missing multiplicative term that is **less than** 1, we specifically have

$$\phi(mn) > \phi(m)\phi(n).$$

- **2.5 8c**: I won't give the answer to this, but I will outline the strategy that makes solving the problem "find all n such that $\phi(n) = x$" a finite process. If $n = p_1^{a_1} \dots p_r^{a_r}$ with $p_1 < p_2 < p_3 \dots < p_r$ then

  $$\phi(n) = n\left(1 - \frac{1}{p_1}\right) \dots \left(1 - \frac{1}{p_r}\right) = p_1^{a_1-1}(p_1 - 1) \dots p_r^{a_r-1}(p_r - 1).$$

  For $\phi(n)$ to equal x, we must have

  $$x = p_1^{a_1-1}(p_1 - 1) \dots p_r^{a_r-1}(p_r - 1).$$

  In particular this means that $(p_1 - 1)(p_2 - 1) \dots (p_r - 1)$ must divide x. There are only finitely many ways to choose $p_1 < p_2 < \dots < p_r$ primes with this happening (no prime larger than x need ever be considered). For each choice of $p_1 < \dots < p_r$ with $(p_1 - 1)(p_2 - 1) \dots (p_r - 1) \leq x$, check does it divide x. If it does, check does the quotient only have $p_1, p_2, \dots p_r$ as factors. If so, say the quotient is $p_1^{b_1} \dots p_r^{b_r}$, then the number

  $$n = p_1^{b_1+1} \dots p_r^{b_r+1}$$

  has $\phi(n) = x$.

  It is clear that it is a finite, if tedious, process to find all such n for a given x

- **2.5 10a**: As we derived in class using inclusion-exclusion,

  $$P_G(q) = \sum_{T \subseteq E(G)} (-1)^{|T|} q^{c(T)}$$

  where $P_G(q)$ is the chromatic polynomial, $E(G)$ is the set of edges of G and $c(T)$ is the number of components in the subgraph created by only looking at the edge set T. Applying this to the yield sign, which has 4 edges, there will be 16 terms in the sum (one for each subset of the edges). We have to look at each subset T and see a) what is the sign of $(-1)^{|T|}$) and b) what is $c(T)$. For example, the three edges that form a triangle have $(-1)^{|T|} = -1$ and $c(T) = 2$, so this subset will contribute $-q^2$ to the sum. Running over all 16 subsets in this way and grouping equal powers we get

  $$p_G(q) = q^4 - 4q^3 + 5q^2 - 2q.$$

  We could also get this directly: there are $q(q-1)(q-2)$ q-colorings of the triangle (as we discussed in class), and each one of these rules out exactly one color for the last vertex, so there are $q(q-1)(q-2)(q-1)$ q-colorings of the whole graph. Expanded out, this is the same as the expression above.

- **2.5 10b**: Using the same technique as for the last part, we find after looking at all 2^6 subsets of edges and gathering terms that

  $$p_G(q) = q^5 - 6q^4 + 15q^3 - 17q^2 + 7q.$$

  Unlike the previous part, there is no quick way to see this.

- **2.5 12a**: $T_0 = 0, T_1 = 0, T_2 = 4, T_3 = 24$

- **2.5 12b**: Without loss of generality suppose that the 6 people are seated as follows: seat 1 has twin 1, seat 2 has twin 2, seat 3 has person 1, seat 4 has person 2, seat 5 has person 3, seat six has person 4. In a valid derangement of the kind required in the question, there are four choice for twin 1 (seats 3 through 6) and subsequently three for twin 2 (the three empty seats from seats 3 through 6 remaining after twin 1 has chosen his seat); so there are 12 choices for twins 1 and 2 together. By symmetry each of these will look the same in terms of how many possibilities there are for seating the remaining people, so assume wlog that twins 1 and 2 occupy seats 5 and 6. We now must distribute persons 1 through 4 among seats 1 through 4, not allowing person 1 to sit in seat 3 or person two to sit in seat 4. By inspection there are 14 ways to do this, so $T_4 = 12 * 14 = 168$.

- **2.5 12c**: The way we counted T_4 suggests a general strategy. Suppose that the $n+2$ people are seated as follows: person i in seat i for $i = 1, \ldots, n$, and twin 1 in seat $n+1$, twin 2 in seat $n+2$. In a valid derangement of the kind required in the question, there are n choices for twin 1 (seats 1 through n) and subsequently $n-1$ for twin 2 (the $n-1$ empty seats from seats 1 through n remaining after twin 1 has chosen his seat); so there are $n(n-1)$ choices for twins 1 and 2 together. By symmetry each of these will look the same in terms of how many possibilities there are for seating the remaining people, so assume wlog that twins 1 and 2 occupy seats 1 and 2. We now must distribute persons 1 through n among seats 3 through $n+2$, not allowing person i to sit in seat i for $i = 3, \ldots, n$ (persons 1 and 2 are ok; they will never sit in their own seats given where twins 1 and 2 went).

  By renaming seats $n+1$ and $n+2$ to be seats 1 and 2, this is the same as rearranging 1 through n in such a way that 3 through n do not go to their original positions, but perhaps 1 and 2 do. Reproducing our count of derangements from class (or page 160 of the textbook) the count turns out to be

  $$n! - \binom{n-2}{1}(n-1)! + \binom{n-2}{2}(n-2)! + \ldots + (-1)^{n-2}2!.$$

  So

  $$T_n = n(n-1)\left(n! - \binom{n-2}{1}(n-1)! + \binom{n-2}{2}(n-2)! + \ldots + (-1)^{n-2}2!\right).$$

  Plugging into Wolfram alpha, I get $T_{10} = 145510740$, which is twice the claimed value. I think that this may be a mistake on the book's part, as the formula above agrees with my computed values of T_0 through T_4.

- **2.5 12d**: The compact (summation form) expression for T_n above is

  $$T_n = \sum_{k=0}^{n-2} n(n-1)(-1)^k \binom{n-2}{k}(n-k)!.$$

So

$$\begin{aligned}\frac{T_n}{(n+2)!} &= \sum_{k=0}^{n-2} \frac{1}{(n+2)!} n(n-1)(-1)^k \binom{n-2}{k}(n-k)! \\ &= \sum_{k=0}^{n-2} (-1)^k \frac{n(n-1)(n-2)!(n-k)!}{k!(n-2-k)!(n+2)!} \\ &= \sum_{k=0}^{n-2} (-1)^k \frac{(n-k)(n-k-1)}{k!(n+2)(n+1)}.\end{aligned}$$

It's not too easy to compute the above limit (one has to do some very careful analysis), but it turns out to be $1/e$.

- For all $n \geq 3$, the chromatic polynomial of C_n is

$$P_{C_n}(q) = (q-1)^n + (-1)^n(q-1).$$

Proof: By inclusion-exclusion we have

$$P_{C_n}(q) = \sum_{S \subseteq E(C_n)} (-1)^{|S|} q^{c(S)}.$$

When $S = \emptyset$ we get q^n. When $|S| = k$ for $k = 1, \ldots, n-1$, the subgraph induced by S is a forest. In a previous exercise we showed that a forest with ℓ components always has $n - \ell$ edges; here $k = n - \ell$, so the forest has $n - k$ components, and so for each possible choice of k edges we get a contribution to the sum of $(-1)^k q^{n-k}$. There are $\binom{n}{k}$ choices of k edges for S; so the contribution to the sum from all S with $|S| < n$ is

$$\sum_{k=0}^{n-1} \binom{n}{k} (-1)^k q^{n-k} = (-1+q)^n - (-1)^n.$$

From $S = E(C_n)$ there is a contribution of $(-1)^n q$, so in all we get

$$P_{C_n}(q) = (-1+q)^n - (-1)^n + (-1)^n q = (q-1)^n + (-1)^n(q-1),$$

as claimed.