Basic Combinatorics

Math 40210, Section 01 — Fall 2012

Homework 9 — Solutions

• 2.6.2 1:

\[
\binom{\alpha-1}{k} + \binom{\alpha-1}{k-1} = \frac{(\alpha-1)(\alpha-2)\ldots(\alpha-k)}{k!} + \frac{(\alpha-1)(\alpha-2)\ldots(\alpha-k+1)}{(k-1)!} \\
= \left(\frac{(\alpha-1)(\alpha-2)\ldots(\alpha-k+1)}{(k-1)!}\right)\left(\frac{\alpha-k}{k} + 1\right) \\
= \left(\frac{(\alpha-1)(\alpha-2)\ldots(\alpha-k+1)}{(k-1)!}\right)\left(\frac{\alpha}{k}\right) \\
= \frac{\alpha(\alpha-1)(\alpha-2)\ldots(\alpha-k+1)}{k!} \\
= \binom{\alpha}{k}.
\]

• 2.6.2 6: In this question we use the basic fact, derived in class and in the textbook, that the number of ways to place \(m\) identical objects into \(n\) distinguishable bins is the same as the number of ways to select \(m\) objects from a set of \(n\) different types of objects with repetition allowed, and in both cases the answer is \(\binom{n+m-1}{m}\).

• 2.6.2 6a: Here we are placing \(m = 50\) identical objects (the burgers) into \(n = 20\) distinguishable bins (the guests), so there are \(\binom{20+50-1}{50} = \binom{69}{50}\) ways.

• 2.6.2 6b: One each guest has received one burger each, there are \(m = 30\) left over, which must be distributed among \(n = 20\) guests, so there are \(\binom{20+30-1}{30} = \binom{49}{30}\) ways.

• 2.6.2 6c: In the first part, there are 51 possibilities for the number \(k\) of burgers left over. If \(k\) are left over, then we are dealing with the problem \(m = 50 - k\) and \(n = 20\), leading to a count of \(\binom{20+50-k-1}{50-k} = \binom{69-k}{50-k}\). So the total count is

\[
\sum_{k=0}^{50} \binom{69-k}{50-k}.
\]

In the second part, again we start by giving each guest one vertex each. Then there are 31 possibilities for the number \(k\) of burgers left over when the balance is distributed. If \(k\) are
left over, then we are dealing with the problem \( m = 30 - k \) and \( n = 20 \), leading to a count of \( \binom{20+30-k-1}{30-k} = \binom{49-k}{30-k} \). So the total count is

\[
\sum_{k=0}^{30} \binom{49-k}{30-k}.
\]

There’s a “trick” way to do both of these parts: introduce a phantom 21st guest to receive the unused burgers. In the first part we are now dealing with the \( m = 50 \), \( n = 21 \) problem, so there are \( \binom{21+50-1}{50} = \binom{70}{50} \) ways. In the second part we are dealing with the \( m = 30 \), \( n = 21 \) problem, so there are \( \binom{21+30-1}{30} = \binom{50}{30} \) ways.

**General comment:** We have just given a combinatorial proof of the following identity: for all \( m, n \),

\[
\sum_{k=0}^{m} \binom{n+m-k-1}{m-k} = \binom{n+m}{m}.
\]

This is identical to

\[
\sum_{k=0}^{m} \binom{n+k-1}{k} = \binom{n+m}{m},
\]

which is a more standard way to present this identity.

• **2.6.4 1(d):** Experiment suggests \( F_{n+1}F_{n-1} - F_n^2 = (-1)^n \). We prove this by induction on \( n \). Base case is \( n = 1 \), which is trivial. We now try to deduce the \( n + 1 \) case from the \( n \) case. The trick is to find exactly the right terms to apply the Fibonacci recurrence to:

\[
F_{(n+1)+1}F_{(n+1)-1} - F_{n+1}^2 = F_{n+2}F_n - F_{n+1}^2
= (F_{n+1} + F_n) F_n - (F_n + F_{n-1}) F_{n+1}
= F_{n+1}F_n + F_n^2 - F_nF_{n+1} - F_{n-1}F_{n+1}
= - (F_{n+1}F_{n-1} - F_{n}^2)
= -(-1)^n \text{ (induction hypothesis)}
= (-1)^{n+1},
\]

as required.

• **2.6.4 3:** Form generating function:

\[
A(x) = 5x + a_2x^2 + a_3x^3 + \ldots
= 5x + (a_1 + 6a_0)x^2 + (a_2 + 6a_1)x^3 + \ldots
= 5x + x(a_1x + a_2x^2 + \ldots) + 6x^2(a_0 + a_1x + \ldots)
= 5x + xA(x) + 6x^2A_x.
\]

So

\[
A(x) = \frac{5x}{1-x-6x^2} = \frac{5x}{(1+2x)(1-3x)} = \frac{1}{1-3x} - \frac{1}{1+2x}
\]

and \( a_n = 3^n - (-1)^n2^n \).
• **2.6.4 5(d):** Induction on \( n \), base case \( n = 0 \) trivial. For the induction step:

\[
\sum_{k=0}^{n+1} F_k^2 = \left( \sum_{k=0}^{n} F_k^2 \right) + F_{n+1}^2
\]

\[
= F_n F_{n+1} + F_{n+1}^2 \quad \text{(induction hypothesis)}
\]

\[
= (F_n + F_{n+1}) F_{n+1}
\]

\[
= F_{n+2} F_{n+1}
\]

\[
= F_{(n+1)+1} F_{n+1},
\]

as required.

• **2.6.4 8(a):** We prove this by induction on \( n \), the base cases \( n = 1 \) and \( 2 \) being trivial. For the induction step:

\[
L_{n+1} = L_n + L_{n-1}
\]

\[
= (F_{n+1} + F_{n-1}) + (F_n + F_{n-2}) \quad \text{(inductive hypothesis)}
\]

\[
= (F_{n+1} + F_n) + (F_{n-1} + F_{n-2})
\]

\[
= F_{n+2} + F_n,
\]

as required. Notice that base cases \( n = 1, 2 \) need to be verified here, since the induction hypothesis is applied both to \( L_n \) and \( L_{n-1} \).

• **2.6.4 10:** Let \( h_n \) be the number of hopscotch boards with \( n \) squares. It’s clear that \( h_0 = 1 \) and \( h_1 = 1 \). For \( n \geq 2 \), there are \( h_{n-1} \) boards that begin with a single-square position (once that square had been put down, it can be completed to a legitimate board by the addition of any \((n-1)\)-square board), and there are \( h_{n-2} \) boards that begin with a two-square position. So \( h_n = h_{n-1} + h_{n-2} \) for \( n \geq 2 \). The \( h \)'s are thus just a “shifted” Fibonacci sequence: \( h_n = F_{n+1} \).

• **2.6.4 11:** From the preceding problem, \( F_n \) is the number of hopscotch boards with \( n - 1 \) squares. How many such Hopscotch boards have exactly \( k \) two-square positions? The \( k \) two-square positions account for \( 2k \) of the squares, leaving \( n - 1 - 2k \) single-square positions, so \((n - 1 - 2k + k = n - k - 1)\) positions in all. To construct an \((n - 1)\)-square Hopscotch board with exactly \( k \) two-square positions, we just select which \( k \) of the \( n - k - 1 \) positions are the two-square ones, so \( \binom{n-k-1}{k} \) choices in all. Summing over all possible \( k \) we get the total number of \((n - 1)\)-square Hopscotch boards:

\[
h_{n-1} = F_n = \sum_k \binom{n-k-1}{k}.
\]

(Practically, \( k \) goes from 0 to the last \( k \) with \( 2k \leq n - 1 \), but for any other \( k \) the binomial coefficient is automatically zero, so we might as well sum over all \( k \).)
• 2.6.5 2(a): Form generating function:

\[ A(x) = a_0 + a_1 x + a_2 x^2 + \ldots \]
\[ = a_0 + (a_0 + c)x + (a_1 + c)x^2 + \ldots \]
\[ = a_0 + x(a_0 + a_1 x + \ldots) + cx(1 + x + x^2 \ldots) \]
\[ = a_0 + xA(x) + \frac{cx}{1 - x}. \]

So

\[ A(x) = \frac{a_0}{1 - x} + \frac{cx}{(1 - x)^2}. \]

The coefficient of \( x^n \) in \( a_0/(1 - x) \) is \( a_0 \) times the coefficient of \( x^n \) in \( 1/(1 - x) \), which is \( a_0 \) times 1 or \( a_0 \). The coefficient of \( x^n \) in \( cx/(1 - x)^2 \) is \( c \) times the coefficient of \( x^n \) in \( x/(1 - x)^2 \), which is \( c \) times \( n \) or \( cn \) (this is equation (2.44) of the book, on page 183). So

\[ a_n = a_0 + cn. \]

• 2.6.5 2(e): Form generating function:

\[ A(x) = a_0 + a_1 x + a_2 x^2 + \ldots \]
\[ = a_0 + (ba_0 + c)x + (ba_1 + 2c)x^2 + (ba_2 + 3c)x^3 + \ldots \]
\[ = a_0 + bx(a_0 + a_1 x + \ldots) + cx(1 + 2x + 3x^2 + \ldots) \]
\[ = a_0 + bxA(x) + \frac{cx}{1 - x} \cdot \frac{1}{1 - x}. \]

(The last part above is obtained either by noticing that the derivative of \( 1/(1 - x) \) is \( 1/(1 - x)^2 \), and that the derivative of \( 1 + x + x^2 + \ldots \), the power series of \( 1/(1 - x) \), is \( 1 + 2x + 3x^2 + \ldots \), so this must be the power series of \( 1/(1 - x)^2 \); or by using equation (2.44) of the text on page 183). So

\[ A(x) = \frac{a_0}{1 - bx} + \frac{cx}{(1 - bx)(1 - x)^2}. \]

We use partial fractions for the second term. Since \( b \neq 1 \) we write

\[ \frac{cx}{(1 - bx)(1 - x)^2} = \frac{A}{1 - bx} + \frac{B}{1 - x} + \frac{C}{(1 - x)^2} \]

and solve to get

\[ A = \frac{bc}{(1 - b)^2}, \quad B = \frac{-c}{1 - b}, \quad C = \frac{1}{1 - b}, \]

so

\[ A(x) = \frac{a_0}{1 - bx} + \left( \frac{bc}{(1 - b)^2} \right) \frac{1}{1 - bx} - \left( \frac{c}{1 - b} \right) \frac{1}{1 - x} + \left( \frac{c}{1 - b} \right) \frac{1}{(1 - x)^2} \]

and

\[ a_n = a_0 b^n + \left( \frac{bc}{(1 - b)^2} \right) b^n + \left( \frac{c}{1 - b} \right) + (n + 1) \left( \frac{c}{1 - b} \right), \]

(the last part using the equation before (2.44) of the book, on page 183).
2.6.5 7(a): $t_0 = 1$, $t_1 = 2$, $t_2 = 4$, $t_3 = 7$ (in this last case, only 111 is left out). For a recurrence: consider how the sequence starts - with a 0, with 10, or with 110 (it can’t start with anything other than these three possibilities). This lets us say

$$t_n = t_{n_1} + t_{n-2} + t_{n-3}.$$ 

We can use this for $n \geq 3$, since it already gives $t_3 = 7$.

2.6.5 7(b): Here’s the generating function of the $t$’s:

$$T(x) = t_0 + t_1 x + t_2 x^2 + t_3 x^3 + t_4 x^4 + \ldots$$

$$= 1 + 2x + 4x^2 + (t_2 + t_1 + t_0)x^3 + (t_3 + t_2 + t_1)x^4 + \ldots$$

$$= 1 + 2x + 4x^2 + x(T(x) - 2x - 1) + x^2(T(x) - 1) + x^3T(x)$$

so

$$T(x) = \frac{1 + x + x^2}{1 - x - x^2 - x^3}.$$ 

2.6.5 7(c): Here’s the generating function of the $t^*$’s:

$$T^*(x) = t^*_0 + t^*_1 x + t^*_2 x^2 + t^*_3 x^3 + \ldots$$

$$= x^2 + t_0 x^3 + t_1 x^4 + \ldots$$

$$= x^2 + x^3T(x)$$

$$= x^2 + \frac{x^3 + x^4 + x^5}{1 - x - x^2 - x^3}$$

$$= \frac{x^2}{1 - x - x^2 - x^3}.$$ 

2.6.6 3(a): $p_3 = 1$, $p_4 = 2$, $p_5 = 5$, $p_0 = 14$. Pictures of $p_3$ through $p_5$ are easy to come up with; for $p_0$, see a picture at http://en.wikipedia.org/wiki/Catalan_number in the section “Applications in Combinatorics”.

2.6.6 3(b): For $p_7$, if the vertices are labeled cyclicly 1 through $n$, and vertex 1 is not an endpoint of one of the the triangulation edges, then there must be an edge from 7 to 2, and there are $p_6$ ways to complete the triangulation.

If 1 is in an edge, and the earliest (in numerical order) vertex that it’s joined to by one of the triangulation edges is 3 then there is one way to complete the triangulation on the 123 side, and $p_6$ ways on the 345671 side.

If 1 is in an edge, and the earliest vertex that it’s joined to by one of the triangulation edges is 4 then there is one way to complete the triangulation on the 1234 side, and $p_5$ ways on the 45671 side.

If 1 is in an edge, and the earliest vertex that it’s joined to by one of the triangulation edges is 5 then there are two ways to complete the triangulation on each side of the 15 edge, independently.
If 1 is in an edge, and the earliest (in numerical order) vertex that it’s joined to by one of the triangulation edges is 6 then there is one way to complete the triangulation on the 671 side, and \( p_5 \) ways on the 123456 side (26 must be an edge, since 1 can’t be joined to 3, 4 or 5).

This gives a total of \( p_6 + p_6 + p_5 + 2 \times 2 + p_5 = 42 \). So \( p_7 = 42 \).

- **2.6.6 3(c):** The last part suggests a general strategy for counting \( p_n \). We look at the earliest vertex (in numerical order) that 1 is joined to by an edge of the triangulation. If that vertex is 3 (the smallest possible) then there is 1 way to complete to triangulation on the 123 side (it’s already completed!) and \( p_{n-1} \) ways on the other side.

If the earliest vertex joined to 1 is \( k \) for some \( n - 1 \leq k > 3 \), then, since 1 cannot be joined to any of 3 through \( k - 1 \), it must be that to triangulate the \( 12 \ldots k \) side we have an edge from 2 to \( k \), leaving \( p_{k-1} \) completions on the polygon 23\ldots\( k \). On the other side (\( k \ldots 1 \)) there are \( p_{n-k+2} \) triangulations (since what’s left is a \( (n - k + 2) \)-sided polygon), and these triangulations can be done independently of the triangulations of the \( 12 \ldots k \) side, giving \( p_{k-1}p_{n-k+2} \) in all.

Finally, if 1 is not an endpoint of one of the the triangulation edges, then there must be an edge from \( n \) to 2, and there are \( p_{n-1} \) ways to complete the triangulation.

We get the recurrence: 

\[
p_3 = 1 \text{ and for } n \geq 3, \quad p_n = p_{n-1} + p_3p_{n-2} + p_4p_{n-3} + \ldots + p_{n-2}p_3 + p_{n-1}.
\]

Defining \( p_2 = 1 \), this can also be written as \( p_2 = 1 \) and for \( n \geq 3, \)

\[
p_n = p_2p_{n-1} + p_3p_{n-2} + p_4p_{n-3} + \ldots + p_{n-2}p_3 + p_{n-1}p_2.
\]

Setting \( p_{n+2} = c_n \), this becomes: \( c_0 = 1 \) and for \( n \geq 1, \)

\[
c_n = c_0c_{n-1} + c_1c_{n-2} + c_2c_{n-3} + \ldots + c_{n-2}c_1 + c_{n-1}c_0.
\]

This is the Catalan recurrence exactly, so

\[
c_n = \binom{2n}{n} \frac{1}{n+1}, \quad p_n = \binom{2n-4}{n-2} \frac{1}{n-1}.
\]

- **2.6.6 5:** If we interpret UP steps as runs scored by White Sox, and DOWN steps as runs scored by Cubs, then a mountain ridgeline is exactly a game between the teams that ends in an \( n\text{-}n \) tie and in which the Cubs never hold the lead, so \( r_n \) is exactly the \( n \)th Catalan number, as we discussed in class.

- **2.6.6 8:** The prime number \( p \) divides \( k! \) exactly

\[
\left[ \frac{k}{p} \right] + \left[ \frac{k}{p^2} \right] + \left[ \frac{k}{p^3} \right] + \ldots
\]

times, where \([x]\) is the greatest integer less than or equal to \( x \). The term \([k/p]\) counts the number of multiples of \( p \) that are at most \( k \); each of these contributes a factor of \( p \); the term
\([k/p^2]\) counts the number of multiples of \(p^2\) that are at most \(k\); each of these contributes a new factor of \(p\) that wasn’t counted in the first term; and so on. Notice that the sum can be thought of as an infinite one: as soon as we get to a term \([k/p^\ell]\) where \(p^\ell\) is greater than \(k\), we just start getting 0’s.

So, for each prime \(p\), the number of times it divides \((2k)!\) is exactly

\[
\left\lfloor \frac{2k}{p} \right\rfloor + \left\lfloor \frac{2k}{p^2} \right\rfloor + \left\lfloor \frac{2k}{p^3} \right\rfloor + \ldots
\]

the number of times it divides \(k!(k+1)!\) is exactly

\[
\left(\left\lfloor \frac{k}{p} \right\rfloor + \left\lfloor \frac{k}{p^2} \right\rfloor + \left\lfloor \frac{k}{p^3} \right\rfloor + \ldots\right) + \left(\left\lfloor \frac{k+1}{p} \right\rfloor + \left\lfloor \frac{k+1}{p^2} \right\rfloor + \left\lfloor \frac{k+1}{p^3} \right\rfloor + \ldots\right).
\]

To show that \((2k)!/(k!(k+1)!))\) is an integer, we need to show that for every prime \(p\), the first expression is at least as big as the second.

It is enough to show that for all integers \(\alpha \geq 1\) and all \(k\) and \(p\) (a prime),

\[
\left\lfloor \frac{2k}{p^\alpha} \right\rfloor \geq \left\lfloor \frac{k}{p^\alpha} \right\rfloor + \left\lfloor \frac{k+1}{p^\alpha} \right\rfloor
\]

Let’s say \(k/p^\alpha = mp^\alpha + r\) where \(0 \leq r \leq p^\alpha - 1\). Then \(2k/p^\alpha = 2mp^\alpha + 2r\) and \((k+1)/p^\alpha = mp^\alpha + r + 1\). We have

\[
\left\lfloor \frac{2k}{p^\alpha} \right\rfloor = \begin{cases} 2m & \text{if } 2r < p^\alpha \\ 2m + 1 & \text{if } 2r \geq p^\alpha, \end{cases}
\]

\[
\left\lfloor \frac{k+1}{p^\alpha} \right\rfloor = \begin{cases} m + 1 & \text{if } r = p^\alpha - 1 \\ m & \text{otherwise}, \end{cases}
\]

and

\[
\left\lfloor \frac{k}{p^\alpha} \right\rfloor = m.
\]

The only way it can happen that

\[
\left\lfloor \frac{2k}{p^\alpha} \right\rfloor < \left\lfloor \frac{k}{p^\alpha} \right\rfloor + \left\lfloor \frac{k+1}{p^\alpha} \right\rfloor
\]

is when \(r = p^\alpha - 1\) and \(2r < p^\alpha\); but this can only happen if \(p^\alpha < 2\), which cannot happen since \(p\) is a prime, so \(\geq 2\), and \(\alpha \geq 1\). So we are done.