1. Define (carefully) $\chi(G)$, the chromatic number of a graph $G$.

**Solution:** The chromatic number of $G$, $\chi(G)$, is the smallest integer $k$ such there exists a coloring of the vertices of $G$ using $k$ colors, with no two adjacent vertices of $G$ receiving the same color.

2. Explain why $\chi(G) \geq n/\alpha(G)$ for every graph $G$, where $\chi(G)$ is the chromatic number of $G$ and $\alpha(G)$ is the size of the largest independent set

**Solution:** Let $K$ be a coloring using $\chi(G)$ colors. Let $C_i$ be the set of vertices colored $i$ by $K$. Since $C_i$ is an independent set, we have $|C_i| \leq \alpha(G)$. Summing over all $i$ gives $n = \sum_i |C_i| \leq \chi(G)\alpha(G)$, which is the same as $n/\alpha(G) \leq \chi(G)$.

3. What is the best that can be said about $\chi(C_7)$, the chromatic number of the 7-cycle, if all we know about chromatic number is the bound from the last part of the question?

**Solution:** $\alpha(C_7) = 3$ (take every second vertex, as long as possible; clearly there is no independent set of size 4), so we can say $\chi(C_7) \geq 7/3 = 2.33$. Since $\chi(C_7)$ is an integer, we can therefore say $\chi(G) \geq 3$. 

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Basic Combinatorics (Math 40210) Sec 01, Fall 2012, Quiz 3

Solutions

October 5, 2012

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