Define a sequence recursively as follows: \( g_0 = 1, g_1 = 2, g_n = g_{n-1} + 2g_{n-2} \) for \( n \geq 2 \).

1. Use induction on \( n \) to show that for all \( n \), \( g_n \geq 2f_n \), where \( f_n \) is the \( n \)th Fibonacci number (defined by the recurrence \( f_0 = 0, f_1 = 1, f_n = f_{n-1} + f_{n-2} \) for \( n \geq 2 \)).

Solution: Base case \( n = 0 \): \( g_0 = 1 \) and \( f_0 = 0; 1 \geq 2 \).

Base case \( n = 1 \): \( g_1 = 2 \) and \( f_1 = 1; 2 \geq 2 \).

Induction step: assuming \( g_k \geq 2f_k \) for all \( k \leq n \), for some \( n \geq 1 \), have (with the first inequality being the induction hypothesis)

\[
\begin{align*}
g_{n+1} &= g_n + 2g_{n-1} \\
&\geq 2f_n + 4f_{n-1} \\
&\geq 2(f_n + f_{n-1}) + 2f_{n-1} \\
&\geq 2(f_n + f_{n-1}) \\
&= 2f_{n+1}.
\end{align*}
\]

2. Find the generating function \( G(x) = g_0 + g_1x + g_2x^2 + \ldots \) of the sequence as an explicit ratio of two polynomials. **THERE’S NO NEED TO FIND AN EXPLICIT EXPRESSION FOR \( g_n \)!**

Solution:

\[
\begin{align*}
G(x) &= g_0 + g_1x + g_2x^2 + g_3x^3 + \ldots \\
&= 1 + 2x + (g_1 + 2g_0)x^2 + (g_2 + 2g_1)x^3 + \ldots \\
&= 1 + 2x + (g_1x^2 + g_2x^3 + \ldots) + (2g_0x^2 + 2g_1x^3 + \ldots) \\
&= 1 + 2x + x(G(x) - g_0) + 2x^2G(x) \\
&= 1 + 2x + x(G(x) - 1) + 2x^2G(x).
\end{align*}
\]

Solving for \( G(x) \):

\[
G(x) = \frac{1 + x}{1 - x - 2x^2}.
\]

This can be simplified (but not necessary for full credit):

\[
\frac{1 + x}{1 - x - 2x^2} = \frac{1 + x}{(1 + x)(1 - 2x)} = \frac{1}{1 - 2x}.
\]

So, it turns out that \( g_n = 2^n \).