Basic Combinatorics (Math 40210) Sec 01, Fall 2012, Quiz 5

Solutions

November 30, 2012

Define a sequence recursively as follows: $g_0 = 1$, $g_1 = 2$, $g_n = g_{n-1} + 2g_{n-2}$ for $n \geq 2$.

1. Use induction on n to show that for all n, $g_n \geq 2f_n$, where f_n is the nth Fibonacci number (defined by the recurrence $f_0 = 0$, $f_1 = 1$, $f_n = f_{n-1} + f_{n-2}$ for $n \geq 2$).

   Solution: Base case $n = 0$: $g_0 = 1$ and $f_0 = 0$; $1 \geq 2.0$.

   Base case $n = 1$: $g_1 = 2$ and $f_1 = 1$; $2 \geq 2.1$. (Notice that since the recurrence is valid only for $n = 2$, I need to verify both $n = 0$ and $n = 1$ to get the induction started.)

   Induction step: assuming $g_k \geq 2f_k$ for all $k \leq n$, for some $n \geq 1$, have (with the first inequality being the induction hypothesis)

$$\begin{aligned} g_{n+1} &= g_n + 2g_{n-1} \\ &\geq 2f_n + 4f_{n-1} \\ &\geq 2(f_n + f_{n-1}) + 2f_{n-1} \\ &\geq 2(f_n + f_{n-1}) \\ &= 2f_{n+1}. \end{aligned}$$

2. Find the generating function $G(x) = g_0 + g_1 x + g_2 x^2 + \ldots$ of the sequence as an explicit ratio of two polynomials. **THERE'S NO NEED TO FIND AN EXPLICIT EXPRESSION FOR g_n!**

   Solution:

$$\begin{aligned} G(x) &= g_0 + g_1 x + g_2 x^2 + g_3 x^3 + \ldots \\ &= 1 + 2x + (g_1 + 2g_0)x^2 + (g_2 + 2g_1)x^3 + \ldots \\ &= 1 + 2x + (g_1 x^2 + g_2 x^3 + \ldots) + (2g_0 x^2 + 2g_1 x^3 + \ldots) \\ &= 1 + 2x + x(g_1 x + g_2 x^2 + \ldots) + 2x^2(g_0 + g_1 x + \ldots) \\ &= 1 + 2x + x(G(x) - g_0) + 2x^2 G(x) \\ &= 1 + 2x + x(G(x) - 1) + 2x^2 G(x). \end{aligned}$$

   Solving for $G(x)$:

$$G(x) = \frac{1+x}{1-x-2x^2}.$$

   This can be simplified (but not necessary for full credit):

$$\frac{1+x}{1-x-2x^2} = \frac{1+x}{(1+x)(1-2x)} = \frac{1}{1-2x}.$$

   So, it turns out that $g_n = 2^n$.