# Basic Combinatorics

## Math 40210, Section 01 — Fall 2012

## Basic graph definitions

It may seem as though the beginning of graph theory comes with a lot of definitions. This may be so, but hopefully by repeatedly using them you will very quickly become accustomed to them all. This document serves as a "cheat-sheet" for all the basic definitions that will be important throughout the course.

- A **graph** (usually denoted by $G$) consists of two objects: a finite, non-empty set $V$ of **vertices**, and a set $E$ of **edges**. The set $V$ can be anything, but the set $E$ must consist of *unordered pairs* of distinct elements of $V$. Think of $V$ as a set of points (called *vertices*), and think of $E$ as specifying which pairs of vertices are joined.

- Properly, what has just been described is a **simple graph**. The *simple* here indicates that we do not allow an edge to go from a vertex to itself, nor do we allow multiple edges to join the same pair of vertices. (Other types of graphs are briefly considered in Section 1.1.1, but will not play a major role in the course.)

- An single element $v$ of $V$ is a called a **vertex**. (NB the unusual singular/plural: vertex/vertices. It's just the same as index/indices, or appendix/appendices.) Properly we should write an edge as $\{u, v\}$ (an unordered set of two distinct vertices), but for the sake of preserving our sanity we will typically just write $uv$, or $e = uv$.

- The edge $e = uv$ is said to **join** $u$ and $v$, and $u$ and $v$, the **endvertices** of $e$, are **adjacent**. Also, both $u$ and $v$ are said to be **incident** with $e$, and we say that $u$ is a **neighbor** of $v$, and vice versa. If $uv$ is *not* an edge, then $u$ and $v$ are **non-adjacent**.

- The **order** of a graph $G$ is the number of vertices, often denoted by $n$, so $n = |V|$. The **size** of $G$ is the number of edges, often denoted by $m$, so $m = |E|$.

- The **open neighborhood** of a vertex $v$, denoted $N(v)$, is the set of vertices that are joined to $v$ by an edge. The **degree** of $v$, denoted $\deg(v)$, is the size of the open neighborhood of $v$, that is, $\deg(v) = |N(v)|$ is the number of neighbors that $v$ has. The **closed neighborhood** of $v$, denoted $N[v]$, consists of $N(v)$ together with $v$ itself. When we say *neighborhood* without any adjective, we always mean the open neighborhood.

- The **degree sequence** of $G$ is the $n$-term sequence listing all of the degrees of vertices of $G$ (most typically in decreasing order). The **maximum degree** of $G$, denoted by

$\Delta$ or $\Delta(G)$, is the largest degree of any vertex in $G$ (so the largest entry in the degree sequence). The **minimum degree**, denoted by $\delta$ or $\delta(G)$, is the smallest degree of any vertex in $G$.

- A **walk** is a list $v_1, \ldots, v_k$ of vertices (not necessarily distinct) with the property that $v_1v_2$, $v_2v_3$, etc., are all edges. If there is no repetition among these edges, then the walk is a **trail**. If there are no repetitions among the vertices, the walk is a **path**. So a *walk* is any kind of continuous perambulation; a *trail* doesn't visit the same edge twice (but may visit a vertex multiple times) and a *path* neither visits an edge nor a vertex twice.

- Adding the edge $v_kv_1$ to a path results in a **closed path** or **cycle** (we will usually say *cycle*). If a trail begins and ends at the same vertex, it is called a **closed trail** or a **circuit** (we will usually say *circuit*).

- The **length** of a walk, trail, path, cycle circuit is the number of edges (counted with repetitions, if necessary) involved.

- A **subgraph** of $G$ is obtained by deleting some (perhaps no) vertices, and then some (perhaps no) edges. When a vertex is deleted, all edges which have that vertex as an endvertex must be deleted too. We write $H \subseteq G$ to indicate that $H$ is a subgraph of $G$, and sometimes say $G$ *contains* $H$. An **induced subgraph** of $G$ is obtained by only deleting some (perhaps no) vertices. Such a subgraph is completely determined by the vertices that we choose not to delete. If $S$ is that set of vertices, we write $\langle S \rangle$ to indicate the induced subgraph. If $H$ is an induced subgraph of $G$, we sometimes say that $G$ *contains* $H$ *as an induced subgraph.*

- The graph on $n$ vertices that has all possible edges is called the **complete graph on** $n$ **vertices**, and it is denoted $K_n$. The graph on $n$ vertices that has no edges is called the **empty graph on** $n$ **vertices**, and it is denoted $E_n$.

- The graph on $n$ vertices $v_1, \ldots, v_n$ that has exactly the edges $v_1v_2, v_2v_3, \ldots, v_{n-1}v_n$ is called the **path on** $n$ **vertices**, and it is denoted $P_n$. If in addition it has the edge $v_nv_1$ then it is called the **cycle on** $n$ **vertices**, and it is denoted $C_n$.

- The **complement** of $G$, denoted $\overline{G}$, is the graph on the same vertex set as $G$ that has $uv$ as an edge if and only if $G$ does not have $uv$ as an edge.

- A graph is said to be **bipartite** is it is possible to partition the vertex set as $V(G) = X \cup Y$ in such a way that all edges have exactly one end vertex in $X$ and one endvertex in $Y$. The sets $X$ and $Y$ (which are not necessarily unique) are called the **partite sets** of $G$, or sometimes the **bipartition classes**. If in a bipartite graph we have all possible edges between $X$ and $Y$ present, then the graph is said to be a **complete bipartite graph**. If $|X| = a$ and $|Y| = b$ then such a graph is denoted $K_{a,b}$.

- A graph is **regular** if every vertex has the same degree. If the degree is given, say $d$, the graph is said to be *d-regular*.

- Two graphs $G$ and $H$ are said to be **isomorphic** if there is a bijection $f : V(G) \to V(H)$ such that $uv \in E(G)$ if and only if $f(u)f(v) \in E(H)$. Another way to say this is that two graphs are isomorphic if they can be made to look identical by a relabeling of the vertices. We write $G \cong H$, or $G = H$, to indicate that $G$ is isomorphic to $H$, and sometimes say that $G$ *is* $H$. If $G$ and $H$ are isomorphic then any statement that is true for $G$ and that does not mention the names of the vertices is also true for $H$. An **unlabeled graph** is a graph in which we do not name the vertices. Strictly speaking, when we talk about an unlabeled graph we are talking simultaneously about a particular labeled graph, and every other graph that is isomorphic to it (for example, there are infinitely many graphs that satisfy the description of a cycle on four vertices; when we talk about *the cycle* on four vertices we mean the whole class of graphs that consist of four vertices and four edges arranged in a cycle, all of which are isomorphic to one another).

- The **distance** between two vertices $u$, $v$ in a graph is the length of the shortest path join them, or $\infty$ if they are not in the same component. It is denoted $d(u, v)$.

- For a connected graph $G$, the **eccentricity** of a vertex $v$, denoted by $\mathrm{ecc}(v)$, is the length of the longest path that starts at $v$. The **radius** of $G$, denoted by $\mathrm{rad}(G)$, is the value of the smallest eccentricity, and the **diameter** of $G$, denoted by $\mathrm{diam}(G)$, is the value of the largest eccentricity. So the diameter is the length of the *longest* path that can be found in $G$.

- The **adjacency matrix** $A = A(G)$ of a graph $G$ with vertices $\{1, \dots, n\}$ is the $n$ by $n$ matrix with $ij$ entry 0 if $ij$ is not an edge, and 1 if it is an edge. The **degree matrix** $D = D(G)$ of $G$ is the $n$ by $n$ matrix with $ii$ entry equal to the degree of vertex $i$ for each $i$, and all other entries 0. The **Laplacian** of $G$ is the $n$ by $n$ matrix $L = D - A$.

- A **tree** is a connected graph that has no cycles (also called an *acyclic* graph). Equivalently, it is a connected graph with $n - 1$ edges (where $n$ is the number of vertices), or an acyclic graph with $n - 1$ edges. A **forest** is a graph with no cycles, or equivalently, a graph all of whose components are trees. A tree is considered to be an example of a forest. A vertex of degree 1 in a tree is called a **leaf**.

- A **spanning tree** of a connected graph is a minimal connected subgraph. Equivalently, it is a subgraph that is a tree, and includes all of the vertices of the graph. If $w : E(G) \to \mathbb{R}^+$ is a function that assigns a non-negative weight $w(e)$ to each edge $e$ of a graph $G$, then the *weight* of a subgraph is the sum of the weights of the edges in that subgraph. A spanning tree that has the property that there is no other spanning tree of smaller weight is called a **minimum weight spanning tree**.

- An **Eulerian trail** (or *Euler trail*) in a graph is a trail that visits every edge (once and only once, by definition of a trail). An **Eulerian circuit** (or *Euler circuit*) is a circuit that visits every edge (and so, by definition of circuit, it begins and ends at the same vertex and visits each edge once and only once). A graph with an Euler *circuit* is said to be *Eulerian.*

- A **Hamiltonian path** (or *Hamilton path*) in a graph is a path that visits every vertex (once and only once, by definition of a path). A **Hamiltonian cycle** (or *Hamilton cycle*) is a cycle that visits every vertex (and so, by definition of cycle, it begins and ends at the same vertex and visits each other vertex once and only once). A graph with a Hamilton path is said to be *traceable*; a graph with a Hamilton cycle is said to be *Hamiltonian* (or a *Hamilton graph*).

- A graph is **planar** if it can be drawn in the plane without any crossing edges, and such a representation is called a *planar representation*. A **region** of a planar representation is a maximal connected piece of the plane, when the drawing of the graph is removed. A **bounding edge** for a region is an edge whose removal increases the area of the region. The **bounding degree** of a region, denoted $b(R)$, is the number of bounding edges that region has.

- A graph $H$ is a **subdivision** of a graph $G$ if it possible to obtain $H$ from $G$ by replacing each edge $uv$ of $G$ with a path, with all the non-end vertices of the paths being new vertices, and with each new vertex being involved in just one path.

- A **proper $k$-coloring** of a graph $G$ is a function $K : V(G) \to \{1, \dots, k\}$ satisfying $K(x) \neq K(y)$ whenever $xy \in E(G)$; in other words, it is an assignment of colors to the vertices, using a palette of $k$ colors, such that adjacent vertices receive distinct colors. A proper $k$-coloring is sometimes just called a *$k$-coloring*. A graph is said to be **$k$-colorable** if there is a $k$-coloring of it. The **chromatic number** of $G$ is the smallest $k$ for which there exists a $k$-coloring. Vertices mapped to the same color in a coloring are called **color classes**.

- A **clique** in a graph $G$ is a set of mutually adjacent vertices (i.e., a complete graph). The **clique number** of $G$, denoted $\omega(G)$, is the number of vertices in the largest clique in $G$. An **independent set** in $G$ is a set of mutually non-adjacent vertices (i.e., an empty graph). The **independence number** of $G$, denoted $\alpha(G)$, is the number of vertices in the largest independent set in $G$.

- A **matching** $M$ in a graph $G$ is a set of disjoint edges, that is, a set of edges that have no end vertices in common. Sometimes such a set of edges is called an *independent set* of edges. If a vertex appears as an endvertex of some edge in $M$, it is said to be **$M$-saturated**; otherwise, it is said to be **$M$-unsaturated**. A matching is **maximal** if it cannot be extended by the addition of more edges, and **maximum** if there is no matching of larger size (i.e., having more edges) in the graph. A matching is said to **perfect** if it saturates all the vertices of the graph.

- For a matching $M$ in a graph $G$, an **$M$-alternating path** is a path in $G$ every second vertex of which lies in $M$. An **$M$-augmenting path** is an $M$-alternating path with the property that the two endvertices of the path are $M$-unsaturated.

- If $G$ is a bipartite graph with partite sets $X$ and $Y$, we say that $X$ can be **matched into** $Y$ if there is a matching in $G$ that saturates $X$, that is, one in which all vertices from $X$ appear as an endvertex of some edge in the matching.

- If $G$ is a bipartite graph with partite sets $X$ and $Y$, the set $X$ is said to **satisfy Hall's condition** if for every $S \subseteq X$, $|N(S)| \geq |S|$, where $N(S)$ is the set of vertices in $Y$ adjacent to something in $S$. If $G$ is an arbitrary graph, it is said to satisfy **Tutte's condition** if for every $S \subseteq V(G)$ we have $\Omega(G - S) \leq |S|$, where $\Omega(H)$ is the number of components of a graph $H$ with odd order.

- For a family $X = \{S_1, \ldots, S_n\}$ of sets, a **system of distinct representatives** is a set $\{x_1, \ldots, x_n\}$ of distinct elements with $x_i \in S_i$ for each $i = 1, \ldots, n$.

- A set $C$ of vertices in a graph G is said to **cover** the edges of $G$ if every edge of $G$ is incident with at least one vertex of $C$. Such a set $C$ is called an **edge cover** of $G$. It is also often called a *vertex cover*.