Using generating functions to solve recurrences
Set up

- Start with recurrence

\[ a_n = c_1 a_{n-1} + \ldots + c_k a_{n-k} \text{ for } n \geq k, \ a_0, \ldots, a_k \text{ given} \]

For example:

\[ f_n = f_{n-1} + f_{n-2} \text{ for } n \geq 2, \ f_0 = 0, \ f_1 = 1 \]

- Form generating function

\[ A(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots \]

For example:

\[ F(x) = f_0 + f_1 x + f_2 x^2 + f_3 x^3 + \ldots + f_n x^n + \ldots \]
Manipulation

- Substitute known values (early on) and recurrence (later)
  For example:

\[
F(x) = 0 + 1x + (f_1 + f_0)x^2 + (f_2 + f_1)x^3 + \ldots + (f_{n-1} + f_{n-2})x^n + \ldots
\]

- Manipulate to get generating function on right-hand side
  For example:

\[
\begin{align*}
F(x) &= x + (f_1 x^2 + f_2 x^3 + \ldots) + (f_0 x^2 + f_1 x^3 + \ldots) \\
&= x + x (f_1 x + f_2 x^2 + \ldots) + x^2 (f_0 + f_1 x + \ldots) \\
&= x + x (F(x) - f_0) + x^2 F(x) \\
&= x + x F(x) + x^2 F(x)
\end{align*}
\]

- Solve for the generating function
  For example:

\[
F(x) = \frac{-x}{x^2 + x - 1}
\]
Solution I

- Find a partial fractions decomposition of the generating function.
  For example:
  \[ F(x) = \frac{A}{x - r_1} + \frac{B}{x - r_2} \]

where

- \( r_1 = \frac{-1 + \sqrt{5}}{2} \)
- \( r_2 = \frac{-1 - \sqrt{5}}{2} \)

are roots of denominator, and

- \( A = -\frac{r_1}{\sqrt{5}} \)
- \( B = \frac{r_2}{\sqrt{5}} \)

are found by combining the fractions, comparing the numerator of the result with the numerator of \( F(x) \), and solving simultaneous equations.
Solution II

- Rewrite the fractions in the form $\frac{1}{1-z}$
  For example:

  \[
  F(x) = \frac{1}{\sqrt{5}} \left( \frac{1}{1 - \frac{x}{r_1}} \right) - \frac{1}{\sqrt{5}} \left( \frac{1}{1 - \frac{x}{r_2}} \right)
  \]

- Find the $n$th term of the sequence by extracting $n$th term of each fraction
  For example:

  \[
  f_n = \frac{1}{\sqrt{5}} \left( \frac{1}{r_1} \right)^n - \frac{1}{\sqrt{5}} \left( \frac{1}{r_2} \right)^n
  \]

- Simplify to taste
  For example:

  \[
  f_n = \frac{1}{\sqrt{5}} \left( \left( \frac{1 + \sqrt{5}}{2} \right)^n + \left( \frac{1 - \sqrt{5}}{2} \right)^n \right)
  \]
Comments

- **Works in principle** for any linear recurrence
  - In practice the partial fractions step can get very tricky, especially if the denominator has repeated roots (and in general the roots of the denominator cannot be found exactly)
- Comparison of closed-form and power series works fine inside radius of convergence of power series (however small)
- Just the closed form for the generating function tells you a lot. If

  \[ A(x) = \frac{p(x)}{q(x)} = \frac{p(x)}{(x - r_1)(x - r_2) \ldots (x - r_k)} \]

  with \( p(x), q(x) \) are polynomials, \( q \) with distinct roots, then

  \[ a_n = \gamma_1 \left( \frac{1}{r_1} \right)^n + \ldots + \gamma_k \left( \frac{1}{r_k} \right)^n \]

  for some constants \( \gamma_1, \ldots, \gamma_k \). So if \( r_1 \) is the closest root to 0,

  \[ a_n \approx \left( \frac{1}{r_1} \right)^n \]
Example: Perrin sequence

- $a_0 = 3$, $a_1 = 0$, $a_2 = 2$ and $a_n = a_{n-2} + a_{n-3}$ for $n \geq 3$
- Solve for $A(X) = a_0 + a_1x + a_2x^2 + \ldots$ to get

\[
\frac{x^2 - 3}{x^3 + x^2 - 1} = \frac{A}{x - r_1} + \frac{B}{x - r_2} + \frac{C}{x - r_3}
\]

where $(x - r_1)(x - r_2)(x - r_3) = x^3 + x^2 - 1$, and, it turns out, $A = -r_1$, $B = -r_2$ and $C = -r_3$.

- So

\[
a_n = \left(\frac{1}{r_1}\right)^n + \left(\frac{1}{r_2}\right)^n + \left(\frac{1}{r_3}\right)^n
\]

and

\[
a_n \approx (1.32471)^n
\]

where 1.32471 \ldots is the **plastic number**
Perrin and primes

- Perrin (1899) noticed:
  - if \( p \) is any prime, then \( p \mid a_p \)
  - if \( n \) is a small composite, then \( n \nmid a_n \)

- He conjectured the following primality test:
  
  \[ p \text{ is a prime if and only if } p \mid a_p \]

- In 1982, Adams and Shanks discovered that
  
  \[ 271441 \mid a_{271441} \left( \approx 10^{33,000} \right) \]

  but \( 271441 = (521)^2 \)

  Numbers like 271441 are called \textit{Perrin pseudoprimes}