Using generating functions to solve recurrences

Math 40210, Fall 2012

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Set up

- Start with recurrence

$$a_n = c_1 a_{n-1} + \ldots c_k a_{n-k} \text{ for } n \geq k,\; a_0, \ldots, a_k \text{ given}$$

For example:

$$f_n = f_{n-1} + f_{n-2} \text{ for } n \geq 2,\; f_0 = 0, f_1 = 1$$

- Form generating function

$$A(x) = a_0 + a_1 x + a_2 x^2 + +a_3 x^3 + \ldots + a_n x^n + \ldots$$

For example:

$$F(x) = f_0 + f_1 x + f_2 x^2 + f_3 x^3 + \ldots + f_n x^n + \ldots$$

Manipulation

- Substitute known values (early on) and recurrence (later)
  For example:

$$F(x) = 0 + 1x + (f_1 + f_0)x^2 + (f_2 + f_1)x^3 + \ldots + (f_{n-1} + f_{n-2})x^n + \ldots$$

- Manipulate to get generating function on right-hand side
  For example:

$$\begin{aligned} F(x) &= x + \left(f_1x^2 + f_2x^3 + \ldots\right) + \left(f_0x^2 + f_1x^3 + \ldots\right) \\ &= x + x\left(f_1x + f_2x^2 + \ldots\right) + x^2\left(f_0 + f_1x + \ldots\right) \\ &= x + x\left(F(x) - f_0\right) + x^2F(x) \\ &= x + xF(x) + x^2F(x) \end{aligned}$$

- Solve for the generating function
  For example:

$$F(x) = \frac{-x}{x^2 + x - 1}$$

Solution I

- Find a partial fractions decomposition of the generating function For example:

$$F(x) = \frac{A}{x - r_1} + \frac{B}{x - r_2}$$

  where
  - $r_1 = \frac{-1+\sqrt{5}}{2}$ and
  - $r_2 = \frac{-1-\sqrt{5}}{2}$

  are roots of denominator, and
  - $A = -r_1/\sqrt{5}$ and
  - $B = r_2/\sqrt{5}$

  are found by combining the fractions, comparing the numerator of the result with the numerator of $F(x)$, and solving simultaneous equations

Solution II

- Rewrite the fractions in the form $\frac{1}{1-z}$
  For example:

$$F(x) = \frac{1}{\sqrt{5}}\left(\frac{1}{1-\frac{x}{r_1}}\right) - \frac{1}{\sqrt{5}}\left(\frac{1}{1-\frac{x}{r_2}}\right)$$

- Find the *n*th term of the sequence by extracting *n*th term of each fraction
  For example:

$$f_n = \frac{1}{\sqrt{5}}\left(\frac{1}{r_1}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1}{r_2}\right)^n$$

- Simplify to taste
  For example:

$$f_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n\right)$$

Comments

- Works *in principle* for any linear recurrence
  In practice the partial fractions step can get very tricky, especially if the denominator has repeated roots (and in general the roots of the denominator cannot be found exactly)
- Comparison of closed-form and power series works fine inside radius of convergence of power series (however small)
- Just the closed form for the generating function tells you a lot. If

  $$A(x) = \frac{p(x)}{q(x)} = \frac{p(x)}{(x - r_1)(x - r_2)\ldots(x - r_k)}$$

  with $p(x)$, $q(x)$ are polynomials, q with distinct roots, then

  $$a_n = \gamma_1 \left(\frac{1}{r_1}\right)^n + \ldots + \gamma_k \left(\frac{1}{r_k}\right)^n$$

  for some constants $\gamma_1, \ldots, \gamma_k$. So if r_1 is the closest root to 0,

  $$a_n \approx \left(\frac{1}{r_1}\right)^n$$

Example: Perrin sequence

- $a_0 = 3$, $a_1 = 0$, $a_2 = 2$ and $a_n = a_{n-2} + a_{n-3}$ for $n \geq 3$
- Solve for $A(X) = a_0 + a_1 x + a_2 x^2 + \ldots$ to get

$$\frac{x^2 - 3}{x^3 + x^2 - 1} = \frac{A}{x - r_1} + \frac{B}{x - r_2} + \frac{C}{x - r_3}$$

where $(x - r_1)(x - r_2)(x - r_3) = x^3 + x^2 - 1$, and, it turns out, $A = -r_1$, $B = -r_2$ and $C = -r_3$.
- So

$$a_n = \left(\frac{1}{r_1}\right)^n + \left(\frac{1}{r_2}\right)^n + \left(\frac{1}{r_3}\right)^n$$

and

$$a_n \approx (1.32471)^n$$

where $1.32471\ldots$ is the *plastic number*

Perrin and primes

- Perrin (1899) noticed:
  - if p is *any* prime, then $p | a_p$
  - if n is a small composite, then $n \not| a_n$
- He conjectured the following primality test:

$$p \text{ is a prime if and only if } p | a_p$$

- In 1982, Adams and Shanks discovered that

$$271441 | a_{271441} \quad \left(\approx 10^{33,000} \right)$$

but $271441 = (521)^2$
Numbers like 271441 are called *Perrin pseudoprimes*