Pascal's triangle

Math 40210, Fall 2012

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Pascal's triangle - symbolic

$$\binom{0}{0}$$
$$\binom{1}{0} \quad \binom{1}{1}$$
$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$
$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$
$$\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4}$$
$$\binom{5}{0} \quad \binom{5}{1} \quad \binom{5}{2} \quad \binom{5}{3} \quad \binom{5}{4} \quad \binom{5}{5}$$
$$\binom{6}{0} \quad \binom{6}{1} \quad \binom{6}{2} \quad \binom{6}{3} \quad \binom{6}{4} \quad \binom{6}{5} \quad \binom{6}{6}$$
$$\binom{7}{0} \quad \binom{7}{1} \quad \binom{7}{2} \quad \binom{7}{3} \quad \binom{7}{4} \quad \binom{7}{5} \quad \binom{7}{6} \quad \binom{7}{7}$$
$$\binom{8}{0} \quad \binom{8}{1} \quad \binom{8}{2} \quad \binom{8}{3} \quad \binom{8}{4} \quad \binom{8}{5} \quad \binom{8}{6} \quad \binom{8}{7} \quad \binom{8}{8}$$
$$\cdots$$

Pascal's triangle - filling out the edges

$$
\begin{array}{c}
1 \\
1 \quad 1 \\
1 \quad \binom{2}{1} \quad 1 \\
1 \quad \binom{3}{1} \quad \binom{3}{2} \quad 1 \\
1 \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad 1 \\
1 \quad \binom{5}{1} \quad \binom{5}{2} \quad \binom{5}{3} \quad \binom{5}{4} \quad 1 \\
1 \quad \binom{6}{1} \quad \binom{6}{2} \quad \binom{6}{3} \quad \binom{6}{4} \quad \binom{6}{5} \quad 1 \\
1 \quad \binom{7}{1} \quad \binom{7}{2} \quad \binom{7}{3} \quad \binom{7}{4} \quad \binom{7}{5} \quad \binom{7}{6} \quad 1 \\
1 \quad \binom{8}{1} \quad \binom{8}{2} \quad \binom{8}{3} \quad \binom{8}{4} \quad \binom{8}{5} \quad \binom{8}{6} \quad \binom{8}{7} \quad 1 \\
\ldots
\end{array}
$$

Pascal's triangle - using formula

$$
\begin{array}{c}
1 \\
1 \quad 1 \\
1 \quad {\color{red}\binom{2}{1}} \quad 1 \\
1 \quad \binom{3}{1} \quad \binom{3}{2} \quad 1 \\
1 \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad 1 \\
1 \quad \binom{5}{1} \quad \binom{5}{2} \quad \binom{5}{3} \quad \binom{5}{4} \quad 1 \\
1 \quad \binom{6}{1} \quad \binom{6}{2} \quad \binom{6}{3} \quad \binom{6}{4} \quad \binom{6}{5} \quad 1 \\
1 \quad \binom{7}{1} \quad \binom{7}{2} \quad \binom{7}{3} \quad \binom{7}{4} \quad \binom{7}{5} \quad \binom{7}{6} \quad 1 \\
1 \quad \binom{8}{1} \quad \binom{8}{2} \quad \binom{8}{3} \quad \binom{8}{4} \quad \binom{8}{5} \quad \binom{8}{6} \quad \binom{8}{7} \quad 1 \\
\cdots
\end{array}
$$

Pascal's triangle - using formula

$$
\begin{array}{c}
1 \\
1 \quad 1 \\
1 \quad {\color{red}1 + 1} \quad 1 \\
1 \quad \binom{3}{1} \quad \binom{3}{2} \quad 1 \\
1 \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad 1 \\
1 \quad \binom{5}{1} \quad \binom{5}{2} \quad \binom{5}{3} \quad \binom{5}{4} \quad 1 \\
1 \quad \binom{6}{1} \quad \binom{6}{2} \quad \binom{6}{3} \quad \binom{6}{4} \quad \binom{6}{5} \quad 1 \\
1 \quad \binom{7}{1} \quad \binom{7}{2} \quad \binom{7}{3} \quad \binom{7}{4} \quad \binom{7}{5} \quad \binom{7}{6} \quad 1 \\
1 \quad \binom{8}{1} \quad \binom{8}{2} \quad \binom{8}{3} \quad \binom{8}{4} \quad \binom{8}{5} \quad \binom{8}{6} \quad \binom{8}{7} \quad 1 \\
\cdots
\end{array}
$$

Pascal's triangle - using Pascal's formula

$$
\begin{array}{c}
1 \\
1 \quad 1 \\
1 \quad 2 \quad 1 \\
1 \quad \binom{3}{1} \quad \binom{3}{2} \quad 1 \\
1 \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad 1 \\
1 \quad \binom{5}{1} \quad \binom{5}{2} \quad \binom{5}{3} \quad \binom{5}{4} \quad 1 \\
1 \quad \binom{6}{1} \quad \binom{6}{2} \quad \binom{6}{3} \quad \binom{6}{4} \quad \binom{6}{5} \quad 1 \\
1 \quad \binom{7}{1} \quad \binom{7}{2} \quad \binom{7}{3} \quad \binom{7}{4} \quad \binom{7}{5} \quad \binom{7}{6} \quad 1 \\
1 \quad \binom{8}{1} \quad \binom{8}{2} \quad \binom{8}{3} \quad \binom{8}{4} \quad \binom{8}{5} \quad \binom{8}{6} \quad \binom{8}{7} \quad 1 \\
\cdots
\end{array}
$$

Pascal's triangle - using Pascal's formula

$$
\begin{array}{c}
1 \\
1 \quad 1 \\
1 \quad 2 \quad 1 \\
1 \quad \color{red}{\binom{3}{1}} \quad \color{red}{\binom{3}{2}} \quad 1 \\
1 \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad 1 \\
1 \quad \binom{5}{1} \quad \binom{5}{2} \quad \binom{5}{3} \quad \binom{5}{4} \quad 1 \\
1 \quad \binom{6}{1} \quad \binom{6}{2} \quad \binom{6}{3} \quad \binom{6}{4} \quad \binom{6}{5} \quad 1 \\
1 \quad \binom{7}{1} \quad \binom{7}{2} \quad \binom{7}{3} \quad \binom{7}{4} \quad \binom{7}{5} \quad \binom{7}{6} \quad 1 \\
1 \quad \binom{8}{1} \quad \binom{8}{2} \quad \binom{8}{3} \quad \binom{8}{4} \quad \binom{8}{5} \quad \binom{8}{6} \quad \binom{8}{7} \quad 1 \\
\cdots
\end{array}
$$

Pascal's triangle - using formula

$$
\begin{array}{c}
1 \\
1 \quad 1 \\
1 \quad 2 \quad 1 \\
1 \quad 1+2 \quad 2+1 \quad 1 \\
1 \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad 1 \\
1 \quad \binom{5}{1} \quad \binom{5}{2} \quad \binom{5}{3} \quad \binom{5}{4} \quad 1 \\
1 \quad \binom{6}{1} \quad \binom{6}{2} \quad \binom{6}{3} \quad \binom{6}{4} \quad \binom{6}{5} \quad 1 \\
1 \quad \binom{7}{1} \quad \binom{7}{2} \quad \binom{7}{3} \quad \binom{7}{4} \quad \binom{7}{5} \quad \binom{7}{6} \quad 1 \\
1 \quad \binom{8}{1} \quad \binom{8}{2} \quad \binom{8}{3} \quad \binom{8}{4} \quad \binom{8}{5} \quad \binom{8}{6} \quad \binom{8}{7} \quad 1 \\
\ldots
\end{array}
$$

Pascal's triangle - using formula

$$
\begin{array}{c}
1 \\
1 \quad 1 \\
1 \quad 2 \quad 1 \\
1 \quad 3 \quad 3 \quad 1 \\
1 \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad 1 \\
1 \quad \binom{5}{1} \quad \binom{5}{2} \quad \binom{5}{3} \quad \binom{5}{4} \quad 1 \\
1 \quad \binom{6}{1} \quad \binom{6}{2} \quad \binom{6}{3} \quad \binom{6}{4} \quad \binom{6}{5} \quad 1 \\
1 \quad \binom{7}{1} \quad \binom{7}{2} \quad \binom{7}{3} \quad \binom{7}{4} \quad \binom{7}{5} \quad \binom{7}{6} \quad 1 \\
1 \quad \binom{8}{1} \quad \binom{8}{2} \quad \binom{8}{3} \quad \binom{8}{4} \quad \binom{8}{5} \quad \binom{8}{6} \quad \binom{8}{7} \quad 1 \\
\ldots
\end{array}
$$

Pascal's triangle - using formula

$$
\begin{array}{c}
1 \\
1 \quad 1 \\
1 \quad 2 \quad 1 \\
1 \quad 3 \quad 3 \quad 1 \\
1 \quad 4 \quad 6 \quad 4 \quad 1 \\
1 \quad \binom{5}{1} \quad \binom{5}{2} \quad \binom{5}{3} \quad \binom{5}{4} \quad 1 \\
1 \quad \binom{6}{1} \quad \binom{6}{2} \quad \binom{6}{3} \quad \binom{6}{4} \quad \binom{6}{5} \quad 1 \\
1 \quad \binom{7}{1} \quad \binom{7}{2} \quad \binom{7}{3} \quad \binom{7}{4} \quad \binom{7}{5} \quad \binom{7}{6} \quad 1 \\
1 \quad \binom{8}{1} \quad \binom{8}{2} \quad \binom{8}{3} \quad \binom{8}{4} \quad \binom{8}{5} \quad \binom{8}{6} \quad \binom{8}{7} \quad 1 \\
\cdots
\end{array}
$$

Pascal's triangle - using formula

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1
...

Pascal's triangle - the row sums

$$
\begin{aligned}
1 &= 1\\
1+1 &= 2\\
1+2+1 &= 4\\
1+3+3+1 &= 8\\
1+4+6+4+1 &= 16\\
1+5+10+10+5+1 &= 32\\
1+6+15+20+15+6+1 &= 64\\
1+7+21+35+35+21+7+1 &= 128\\
1+8+28+56+70+56+28+8+1 &= 256\\
&\dots
\end{aligned}
$$

Pascal's triangle - the symbolic row sums

$$\binom{0}{0} = 1$$
$$\binom{1}{0} + \binom{1}{1} = 2$$
$$\binom{2}{0} + \binom{2}{1} + \binom{2}{2} = 4$$
$$\binom{3}{0} + \binom{3}{1} + \binom{3}{2} + \binom{3}{3} = 8$$
$$\binom{4}{0} + \binom{4}{1} + \binom{4}{2} + \binom{4}{3} + \binom{4}{4} = 16$$
$$\binom{5}{0} + \binom{5}{1} + \binom{5}{2} + \binom{5}{3} + \binom{5}{4} + \binom{5}{5} = 32$$
$$\binom{6}{0} + \binom{6}{1} + \binom{6}{2} + \binom{6}{3} + \binom{6}{4} + \binom{6}{5} + \binom{6}{6} = 64$$
$$\binom{7}{0} + \binom{7}{1} + \binom{7}{2} + \binom{7}{3} + \binom{7}{4} + \binom{7}{5} + \binom{7}{6} + \binom{7}{7} = 128$$
$$\binom{8}{0} + \binom{8}{1} + \binom{8}{2} + \binom{8}{3} + \binom{8}{4} + \binom{8}{5} + \binom{8}{6} + \binom{8}{7} + \binom{8}{8} = 256$$
$$\cdots$$

Identity: for all $n \geq 0$,

$$\sum_{i=0}^{n} \binom{n}{i} = 2^n$$

Pascal's triangle - the alternating row sums

$$
\begin{gathered}
1 \; = 1 \\
1 - 1 \; = 0 \\
1 - 2 + 1 \; = 0 \\
1 - 3 + 3 - 1 \; = 0 \\
1 - 4 + 6 - 4 + 1 \; = 0 \\
1 - 5 + 10 - 10 + 5 - 1 \; = 0 \\
1 - 6 + 15 - 20 + 15 - 6 + 1 \; = 0 \\
1 - 7 + 21 - 35 + 35 - 21 + 7 - 1 \; = 0 \\
1 - 8 + 28 - 56 + 70 - 56 + 28 - 8 + 1 \; = 0 \\
\ldots
\end{gathered}
$$

Pascal's triangle - the symbolic alternating row sums

$$\binom{0}{0} = 1$$
$$\binom{1}{0} - \binom{1}{1} = 0$$
$$\binom{2}{0} - \binom{2}{1} + \binom{2}{2} = 0$$
$$\binom{3}{0} - \binom{3}{1} + \binom{3}{2} - \binom{3}{3} = 0$$
$$\binom{4}{0} - \binom{4}{1} + \binom{4}{2} - \binom{4}{3} + \binom{4}{4} = 0$$
$$\binom{5}{0} - \binom{5}{1} + \binom{5}{2} - \binom{5}{3} + \binom{5}{4} - \binom{5}{5} = 0$$
$$\binom{6}{0} - \binom{6}{1} + \binom{6}{2} - \binom{6}{3} + \binom{6}{4} - \binom{6}{5} + \binom{6}{6} = 0$$
$$\binom{7}{0} - \binom{7}{1} + \binom{7}{2} - \binom{7}{3} + \binom{7}{4} - \binom{7}{5} + \binom{7}{6} - \binom{7}{7} = 0$$
$$\binom{8}{0} - \binom{8}{1} + \binom{8}{2} - \binom{8}{3} + \binom{8}{4} - \binom{8}{5} + \binom{8}{6} - \binom{8}{7} + \binom{8}{8} = 0$$
$$\cdots$$

Identity: for all $n \geq 1$,

$$\sum_{i=0}^{n} (-1)^i \binom{n}{i} = 0$$

Pascal's triangle - summing along diagonals

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1
...

Pascal's triangle - summing along diagonals

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1
. . .

Pascal's triangle - summing along diagonals

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1
. . .

Pascal's triangle - summing along diagonals

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1
...

Pascal's triangle - symbolic sum along diagonals

$$\binom{0}{0}$$
$$\binom{1}{0} \quad \binom{1}{1}$$
$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$
$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$
$$\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4}$$
$$\binom{5}{0} \quad \binom{5}{1} \quad \binom{5}{2} \quad \binom{5}{3} \quad \binom{5}{4} \quad \binom{5}{5}$$
$$\binom{6}{0} \quad \binom{6}{1} \quad \binom{6}{2} \quad \binom{6}{3} \quad \binom{6}{4} \quad \binom{6}{5} \quad \binom{6}{6}$$
$$\binom{7}{0} \quad \binom{7}{1} \quad \binom{7}{2} \quad \binom{7}{3} \quad \binom{7}{4} \quad \binom{7}{5} \quad \binom{7}{6} \quad \binom{7}{7}$$
$$\binom{8}{0} \quad \binom{8}{1} \quad \binom{8}{2} \quad \binom{8}{3} \quad \binom{8}{4} \quad \binom{8}{5} \quad \binom{8}{6} \quad \binom{8}{7} \quad \binom{8}{8}$$
$$\cdots$$

Identity: for all $k \geq 0$, and $s \geq k$

$$\sum_{i=k}^{s} \binom{i}{k} = \binom{s+1}{k+1}$$

The vandermonde convolution

- Each week, IN lottery selects m numbers from a bag with $m+n$ numbers. When you buy a lottery ticket, you select ℓ numbers from the $m+n$. You have a *k-win* if you have exactly k of the the selected numbers among your ℓ.
- How many tickets are k-wins?
- It's $\binom{m}{k}\binom{n}{\ell-k}$ (automatically 0 if $k>\ell$ or $k>m$)
- How many tickets in all?

$$\binom{m+n}{\ell}=\sum_{k}\binom{m}{k}\binom{n}{\ell-k}\ \left(\text{or}\ \sum_{k=0}^{\min\{m,\ell\}}\binom{m}{k}\binom{n}{\ell-k}\right)$$