

Pascal's triangle

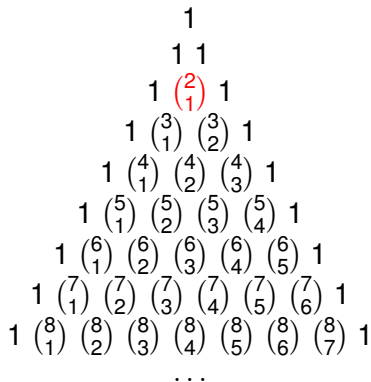
Math 40210, Fall 2012

October 25, 2012

Pascal's triangle - filling out the edges

$$\begin{array}{cccccccc} & & & & & & & 1 \\ & & & & & & & 1 & 1 \\ & & & & & & 1 & \binom{2}{1} & 1 \\ & & & & & 1 & \binom{3}{1} & \binom{3}{2} & 1 \\ & & & & 1 & \binom{4}{1} & \binom{4}{2} & \binom{4}{3} & 1 \\ & & & 1 & \binom{5}{1} & \binom{5}{2} & \binom{5}{3} & \binom{5}{4} & 1 \\ & & 1 & \binom{6}{1} & \binom{6}{2} & \binom{6}{3} & \binom{6}{4} & \binom{6}{5} & 1 \\ & 1 & \binom{7}{1} & \binom{7}{2} & \binom{7}{3} & \binom{7}{4} & \binom{7}{5} & \binom{7}{6} & 1 \\ 1 & \binom{8}{1} & \binom{8}{2} & \binom{8}{3} & \binom{8}{4} & \binom{8}{5} & \binom{8}{6} & \binom{8}{7} & 1 \\ & & & & & & & & & \dots \end{array}$$

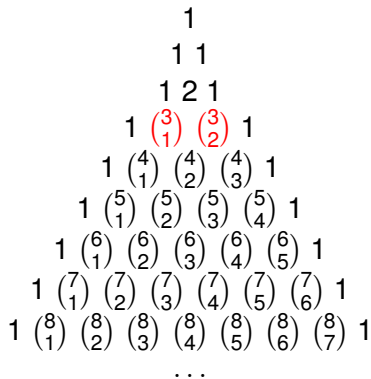
Pascal's triangle - using formula



Pascal's triangle - using Pascal's formula

$$\begin{array}{cccccccc} & & & & 1 & & & & \\ & & & & 1 & 1 & & & \\ & & & 1 & 2 & 1 & & & \\ & & 1 & \binom{3}{1} & \binom{3}{2} & 1 & & & \\ & 1 & \binom{4}{1} & \binom{4}{2} & \binom{4}{3} & 1 & & & \\ & 1 & \binom{5}{1} & \binom{5}{2} & \binom{5}{3} & \binom{5}{4} & 1 & & \\ & 1 & \binom{6}{1} & \binom{6}{2} & \binom{6}{3} & \binom{6}{4} & \binom{6}{5} & 1 & \\ & 1 & \binom{7}{1} & \binom{7}{2} & \binom{7}{3} & \binom{7}{4} & \binom{7}{5} & \binom{7}{6} & 1 \\ 1 & \binom{8}{1} & \binom{8}{2} & \binom{8}{3} & \binom{8}{4} & \binom{8}{5} & \binom{8}{6} & \binom{8}{7} & 1 \\ & & & & \dots & & & & \end{array}$$

Pascal's triangle - using Pascal's formula



Pascal's triangle - using formula

$$\begin{array}{cccccccc} & & & & 1 & & & & \\ & & & & 1 & 1 & & & \\ & & & 1 & 2 & 1 & & & \\ & & 1 & 1 & + & 2 & 2 & + & 1 & 1 & \\ & 1 & \binom{4}{1} & \binom{4}{2} & \binom{4}{3} & 1 & & & & & \\ & 1 & \binom{5}{1} & \binom{5}{2} & \binom{5}{3} & \binom{5}{4} & 1 & & & & \\ & 1 & \binom{6}{1} & \binom{6}{2} & \binom{6}{3} & \binom{6}{4} & \binom{6}{5} & 1 & & & \\ & 1 & \binom{7}{1} & \binom{7}{2} & \binom{7}{3} & \binom{7}{4} & \binom{7}{5} & \binom{7}{6} & 1 & & \\ 1 & \binom{8}{1} & \binom{8}{2} & \binom{8}{3} & \binom{8}{4} & \binom{8}{5} & \binom{8}{6} & \binom{8}{7} & 1 & & \\ & & & & \dots & & & & & & \end{array}$$

Pascal's triangle - using formula

$$\begin{array}{cccccccc} & & & & 1 & & & & \\ & & & & 1 & 1 & & & \\ & & & 1 & 2 & 1 & & & \\ & & 1 & 3 & 3 & 1 & & & \\ & 1 & \binom{4}{1} & \binom{4}{2} & \binom{4}{3} & 1 & & & \\ & 1 & \binom{5}{1} & \binom{5}{2} & \binom{5}{3} & \binom{5}{4} & 1 & & \\ & 1 & \binom{6}{1} & \binom{6}{2} & \binom{6}{3} & \binom{6}{4} & \binom{6}{5} & 1 & \\ & 1 & \binom{7}{1} & \binom{7}{2} & \binom{7}{3} & \binom{7}{4} & \binom{7}{5} & \binom{7}{6} & 1 \\ 1 & \binom{8}{1} & \binom{8}{2} & \binom{8}{3} & \binom{8}{4} & \binom{8}{5} & \binom{8}{6} & \binom{8}{7} & 1 \\ & & & & \dots & & & & \end{array}$$

Pascal's triangle - using formula

$$\begin{array}{cccccccc} & & & & 1 & & & & \\ & & & & 1 & 1 & & & \\ & & & 1 & 2 & 1 & & & \\ & & 1 & 3 & 3 & 1 & & & \\ & 1 & 4 & 6 & 4 & 1 & & & \\ & 1 & \binom{5}{1} & \binom{5}{2} & \binom{5}{3} & \binom{5}{4} & 1 & & \\ & 1 & \binom{6}{1} & \binom{6}{2} & \binom{6}{3} & \binom{6}{4} & \binom{6}{5} & 1 & \\ & 1 & \binom{7}{1} & \binom{7}{2} & \binom{7}{3} & \binom{7}{4} & \binom{7}{5} & \binom{7}{6} & 1 \\ 1 & \binom{8}{1} & \binom{8}{2} & \binom{8}{3} & \binom{8}{4} & \binom{8}{5} & \binom{8}{6} & \binom{8}{7} & 1 \\ & & & & \dots & & & & \end{array}$$

Pascal's triangle - using formula

$$\begin{array}{cccccccc} & & & & 1 & & & & \\ & & & & & & & & \\ & & & & & & & & \\ & & & & 1 & 1 & & & \\ & & & & 1 & 2 & 1 & & \\ & & & & 1 & 3 & 3 & 1 & \\ & & & & 1 & 4 & 6 & 4 & 1 \\ & & & & 1 & 5 & 10 & 10 & 5 & 1 \\ & & & & 1 & 6 & 15 & 20 & 15 & 6 & 1 \\ & & & & 1 & 7 & 21 & 35 & 35 & 21 & 7 & 1 \\ & & & & 1 & 8 & 28 & 56 & 70 & 56 & 28 & 8 & 1 \\ & & & & & & & & & & & & \\ & & & & & & & & & & & & \dots \end{array}$$

Pascal's triangle - the row sums

$$1 = 1$$

$$1 + 1 = 2$$

$$1 + 2 + 1 = 4$$

$$1 + 3 + 3 + 1 = 8$$

$$1 + 4 + 6 + 4 + 1 = 16$$

$$1 + 5 + 10 + 10 + 5 + 1 = 32$$

$$1 + 6 + 15 + 20 + 15 + 6 + 1 = 64$$

$$1 + 7 + 21 + 35 + 35 + 21 + 7 + 1 = 128$$

$$1 + 8 + 28 + 56 + 70 + 56 + 28 + 8 + 1 = 256$$

...

Pascal's triangle - the symbolic row sums

$$\begin{aligned} & \binom{0}{0} = 1 \\ & \binom{1}{0} + \binom{1}{1} = 2 \\ & \binom{2}{0} + \binom{2}{1} + \binom{2}{2} = 4 \\ & \binom{3}{0} + \binom{3}{1} + \binom{3}{2} + \binom{3}{3} = 8 \\ & \binom{4}{0} + \binom{4}{1} + \binom{4}{2} + \binom{4}{3} + \binom{4}{4} = 16 \\ & \binom{5}{0} + \binom{5}{1} + \binom{5}{2} + \binom{5}{3} + \binom{5}{4} + \binom{5}{5} = 32 \\ & \binom{6}{0} + \binom{6}{1} + \binom{6}{2} + \binom{6}{3} + \binom{6}{4} + \binom{6}{5} + \binom{6}{6} = 64 \\ & \binom{7}{0} + \binom{7}{1} + \binom{7}{2} + \binom{7}{3} + \binom{7}{4} + \binom{7}{5} + \binom{7}{6} + \binom{7}{7} = 128 \\ & \binom{8}{0} + \binom{8}{1} + \binom{8}{2} + \binom{8}{3} + \binom{8}{4} + \binom{8}{5} + \binom{8}{6} + \binom{8}{7} + \binom{8}{8} = 256 \\ & \dots \end{aligned}$$

Identity: for all $n \geq 0$,

$$\sum_{i=0}^n \binom{n}{i} = 2^n$$

Pascal's triangle - the alternating row sums

$$1 = 1$$

$$1 - 1 = 0$$

$$1 - 2 + 1 = 0$$

$$1 - 3 + 3 - 1 = 0$$

$$1 - 4 + 6 - 4 + 1 = 0$$

$$1 - 5 + 10 - 10 + 5 - 1 = 0$$

$$1 - 6 + 15 - 20 + 15 - 6 + 1 = 0$$

$$1 - 7 + 21 - 35 + 35 - 21 + 7 - 1 = 0$$

$$1 - 8 + 28 - 56 + 70 - 56 + 28 - 8 + 1 = 0$$

...

Pascal's triangle - the symbolic alternating row sums

$$\begin{array}{c} \binom{0}{0} = 1 \\ \binom{1}{0} - \binom{1}{1} = 0 \\ \binom{2}{0} - \binom{2}{1} + \binom{2}{2} = 0 \\ \binom{3}{0} - \binom{3}{1} + \binom{3}{2} - \binom{3}{3} = 0 \\ \binom{4}{0} - \binom{4}{1} + \binom{4}{2} - \binom{4}{3} + \binom{4}{4} = 0 \\ \binom{5}{0} - \binom{5}{1} + \binom{5}{2} - \binom{5}{3} + \binom{5}{4} - \binom{5}{5} = 0 \\ \binom{6}{0} - \binom{6}{1} + \binom{6}{2} - \binom{6}{3} + \binom{6}{4} - \binom{6}{5} + \binom{6}{6} = 0 \\ \binom{7}{0} - \binom{7}{1} + \binom{7}{2} - \binom{7}{3} + \binom{7}{4} - \binom{7}{5} + \binom{7}{6} - \binom{7}{7} = 0 \\ \binom{8}{0} - \binom{8}{1} + \binom{8}{2} - \binom{8}{3} + \binom{8}{4} - \binom{8}{5} + \binom{8}{6} - \binom{8}{7} + \binom{8}{8} = 0 \\ \dots \end{array}$$

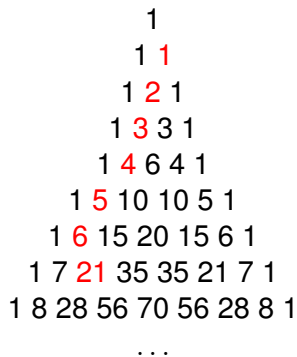
Identity: for all $n \geq 1$,

$$\sum_{i=0}^n (-1)^i \binom{n}{i} = 0$$

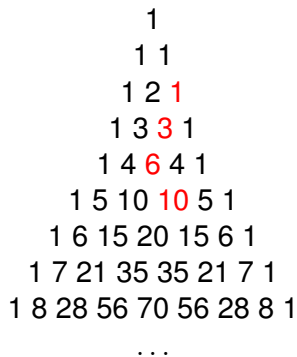
Pascal's triangle - summing along diagonals

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1
...

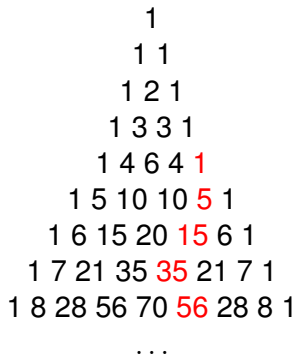
Pascal's triangle - summing along diagonals



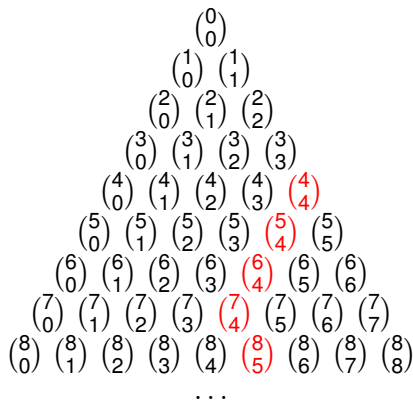
Pascal's triangle - summing along diagonals



Pascal's triangle - summing along diagonals



Pascal's triangle - symbolic sum along diagonals



Identity: for all $k \geq 0$, and $s \geq k$

$$\sum_{i=k}^s \binom{i}{k} = \binom{s+1}{k+1}$$

The vandermonde convolution

- Each week, IN lottery selects m numbers from a bag with $m + n$ numbers. When you buy a lottery ticket, you select ℓ numbers from the $m + n$. You have a k -win if you have exactly k of the the selected numbers among your ℓ .
- How many tickets are k -wins?
- It's $\binom{m}{k} \binom{n}{\ell-k}$ (automatically 0 if $k > \ell$ or $k > m$)
- How many tickets in all?

$$\binom{m+n}{\ell} = \sum_k \binom{m}{k} \binom{n}{\ell-k} \left(\text{or } \sum_{k=0}^{\min\{m,\ell\}} \binom{m}{k} \binom{n}{\ell-k} \right)$$