Prüfer codes and Cayley’s formula

Math 40210, Fall 2012

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The Prüfer code of a tree

The Construction

- Given: a tree $T$ on vertex set $\{v_1, \ldots, v_n\}$, with $v_1 < v_2 \ldots < v_n$
- Repeat until only one edge left:
  - Delete leaf with lowest label (result is smaller tree)
  - Record the label of the deleted leaf’s unique neighbor
- Result is a string of length $n - 2$ on alphabet $\{v_1, \ldots, v_n\}$, the Prüfer code of $T$

Some Facts

- FACT 1: Each vertex $v_i$ appears $d(v_i) - 1$ times in the Prüfer code
- FACT 2: Once first leaf (say $v_i$) is deleted, and first label (say $v_j$) recorded, rest of Prüfer code is exactly the Prüfer code of $T - v_i$ on vertex set $\{v_1, \ldots, v_n\} \setminus \{v_i\}$
Different trees have different Prüfer codes

Proof by Induction

- Case $n = 3$ easily verified
- For $n \geq 4$: given trees $T_1, T_2$ on $\{v_1, \ldots, v_n\}$
  - IF lowest-labeled leaves are different, then the Prüfer codes are different (by FACT 1)
  - IF lowest-labeled leaves the same, but labels of unique neighbours different, THEN the Prüfer codes are different (by construction)
  - IF lowest-labeled leaves the same, labels of unique neighbours the same, THEN step one of Prüfer code constructions agree; but resulting smaller trees are different, so have different Prüfer codes (by induction)

Map from Trees to Prüfer Codes is *Injective*
Every string is the Prüfer code of some tree

Proof by Induction

- Case $n = 3$ easily verified
- For $n \geq 4$: given string $S = (\sigma_1, \ldots, \sigma_{n-2})$ on alphabet $\{v_1, \ldots, v_n\}$
  - Find the lowest labeled vertex that does not appear in the string, $v_i$ say
  - Form $S' = (\sigma_2, \ldots, \sigma_{n-2})$ by deleting first entry from $S$
  - $S'$ is a string of length $n - 3$ on alphabet $\{v_1, \ldots, v_n\} \setminus v_i$, so (by induction) there is a tree $T'$ on $\{v_1, \ldots, v_n\} \setminus v_i$ with $S'$ as its Prüfer code
  - Form $T$ from $T'$ by adding vertex $v_i$, joined only to $\sigma_1$
  - Prüfer code of $T$ starts $\sigma_1$ and (by FACT 2) continues with $S'$, so is $S$

Map from Trees to Prüfer Codes is Surjective, so BIJECTIVE

Cayley’s Formula:

There are exactly $n^{n-2}$ labelled trees on $n$ vertices
The tree of a Prüfer code

Unraveling the Induction

- Given: a string $S$ of length $n - 2$ on alphabet $\{v_1, \ldots, v_n\}$, with $v_1 < v_2 \ldots < v_n$
- Repeat until $S$ is empty and alphabet has size 2:
  - Identify the lowest letter in the alphabet that does not appear in the string, $v_i$ say, and the first element of the string, $v_j$ say
  - Add $v_i$ to the graph being constructed (if it isn’t already there), and join it to $v_j$ (adding $v_j$ to the graph first if necessary)
  - Remove $v_i$ from the alphabet, and remove the first term from the string
- Join the two remaining vertices in the alphabet
- Result is a tree on vertex set $\{v_1, \ldots, v_n\}$ with Prüfer code $S$