Math 40210, Basic Combinatorics, Fall 2015

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Abstract
This document includes
• homeworks, quizzes and exams, and
• supplementary notes
for the Fall 2015 incarnation of Math 40210 — Basic Combinatorics, an undergraduate course offered by the Department of Mathematics at the University of Notre Dame. The notes have been written in a single pass, and as such may well contain typographical (and sometimes more substantial) errors. Comments and corrections will be happy received at dgalvin1@nd.edu.

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1 Introduction

Combinatorics is the study of finite or countable discrete structures. “Discrete” here means that the structures are made up of distinct and separated parts (in contrast to “continuous” structures such as the real line). Combinatorics is an area of mathematics that has been increasing in importance in recent decades, in part because of the advent of digital computers (which operate and store data discretely), and in part because of the recent ubiquity of large discrete networks (social, biological, ecological, ...).

Typical objects studied in combinatorics include permutations (arrangements of distinct objects in various different orders), graphs (networks consisting of nodes, some pairs of which are joined), and finite sets and their subsets.

There are many subfields of combinatorics, such as enumerative (e.g., in how many ways can $n$ objects in a row be rearranged, such that no object is returned to its original position?), structural (e.g., when is it possible to travel around a network, visiting each edge once and only once?), and extremal (e.g., what’s the largest number of subsets of a set of size $n$ that can be chosen in such a way that any two of them have at least one element in common?). In this course, we will explore each of these aspects of combinatorics, and maybe some more as time permits.

Rather than trying to give a cogent definition of combinatorics here, I’ll try instead to give a flavor of the subject by mentioning some famous the problems studied in the field,
many of which we will solve to some degree over the course of the semester.

1.1 The bridges of Königsberg

The town of Kaliningrad, Russia is built around two large islands on the River Pregel. The map below shows how the two islands (A and D) and the north and south banks (C and B) were connected by seven bridges in 1736 (at which time the town was called Königsberg, and was located in Prussia).

A popular puzzle among residents of Königsberg at that time was: is it possible to walk around the town in a single pass, crossing each bridge once and only once?

Leonhard Euler’s 1736 resolution of this puzzle is often considered the first research in what we now call “graph theory”, a major branch of combinatorics. Euler addressed the more general question: given any collection of rivers, islands, banks and bridges, is there a simple way of determining whether it is possible to walk around the configuration in a single pass, crossing each bridge once and only once? He gave a complete answer to the question.

1.2 The utilities problem

There are three houses and three utility companies (water, gas, electric). Is it possible to connect each of the three houses to each of the three utilities without having any of the connections cross? (The figure below shows a solution with one crossing connection).
More generally, given a collection of nodes that have to be joined up into a network by making connections between certain (predetermined) pairs of the nodes, is it possible to create the desired connections in such a way that no two of them cross? Is there a quick way to recognize when this is possible? When it is not possible, what is the configuration of connections that minimizes that number of crossing? These are all questions that have been studied in graph theory, with varying degrees of success. For example, we know definitively that if there are nine houses and nine utilities then it is not possible to connect the houses to the utilities with a crossed connection, but we do not know what configuration gives the least number of crossings, nor what that least number is (it is believed to be 256).

1.3 The coat-check problem

46 absent-minded professors go to the theater, and each leaves their coat at the coat-check. But (being absent-minded), by the time the show is over they have all lost their cost-check tickets. They decide to solve the problem of retrieving coats at the end of the show by each selecting a coat at random.

How likely is it that none of the professors goes home with her own coat? What would the answer be if there were \(n\) professors, rather than 46?

1.4 The Catalan numbers

1. 14 people sit around a circular table. In how many ways can they pair off the shake hands in such a way that there are no pairs of crossing hands? (For example, my neighbor to my right can’t pair with my neighbor to the left, because that would leave
me with nobody to pair with — whoever I choose as my partner, our handshake will
cross hands with the handshake of my two neighbors).

2. In how many different ways can the Cubs and the White Sox play to a 7-7 tie, with the
Cubs never trailing? (Here I consider two games to be different if the orders in which
the runs are scored are different: a game in which the Cubs score seven runs to take a
7-0 lead, then the White Sox score seven, is different from a game in which the Cubs
score six, then the White Sox score one, then the Cubs score their seventh, then the
White Sox score six).

3. In how many ways can a convex 16-gon be triangulated (broken up into triangles by
the drawing of non-crossing diagonals)?

4. Why do the previous three questions all have the same answer????

1.5 Minimum spanning tree

Up to this point the questions may have seemed a little frivolous. Here is a problem of great
practical import.

A group of \( n \) cities wants to set up a network of fiberoptic cables, in such a way that it is
possible to send information from any one city to any other along the network (perhaps via
intermediate cities).

An engineering firm has determined, for each pair of cities, the cost of laying a cable
directly between those two cities.

What pairs of cities should have cable laid directly between them, to create a minimum-
cost fully connected network.

Brute force does not help here. Even after making the intelligent observation that a
minimum-cost network will not contain a cycle of cities (e.g., cities A, B, C and D with A
connected directly to B, B to C, C to A and A to D) — since if it did, a cheaper and still
fully-connected network would be obtainable by deleting any one of the connections along
the cycle — there are still an enormous number of candidate fully-connected networks. For
example, when \( n = 100 \), there are \( 10^{196} \) cycle-free fully connected networks to consider (we
will verify that this count is correct during the semester).

We will give an answer to this question that can be used to quickly determine a minimum-
cost fully-connected network, not only when \( n = 100 \) but even when \( n \) is as large as 100,000!
2 Test your intuition

1. How many 6-character license plates can be made using letters and numbers (36 available characters in all)?

   (a) $36 \times 35 \times 34 \times 33 \times 32 \times 31$
   (b) $6^{36}$
   (c) $36 \times 6$
   (d) $36^6$

2. There were 11 different styles of “York, ME” fridge-magnets for sale at a beach shop last Saturday. I wanted to buy magnets for each of 4 different colleagues. In how many ways could I have done this?

   (a) $11^4$
   (b) $4^{11}$
   (c) $\binom{11}{4}$
   (d) 44

3. Back to the license plates ... how many have no repeated characters?

   (a) $6!$
   (b) $\binom{36}{6}$
   (c) $36 \times 35 \times 34 \times 33 \times 32 \times 31$
   (d) $36^6$
3 Homework 1 — due Thursday September 3

General homework information

In general homework will be assigned on a Thursday, and will be due in class the following Tuesday (this homework is an exception, as it will be due on Thursday). As they are assigned, homework problems will be added to this document, and it will be posted on the website. At the same time I will make an email announcement.

I will distribute a printout of the homework problems in class each week, with plenty of blank space. The solutions that you submit must be presented on this printout (using the flip side of a page if necessary). You may also print out the relevant pages from the website; if you do so, you must staple your pages together. I reserve the right to

- not grade any homework that is not presented on the printout, and
- deduct points for homework that is not stapled.

The weekly homework is an important part of the course; it gives you a chance to think more deeply about the material, and to go from seeing (in lectures) to doing. It’s also your opportunity to show me that you are engaging with the course topics.

Homework is an essential part of your learning in this course, so please take it very seriously. It is extremely important that you keep up with the homework, as if you do not, you may quickly fall behind in class and find yourself at a disadvantage during exams.

You should treat the homework as a learning opportunity, rather than something you need to get out of the way. Reread, revise, and polish your solutions until they are correct, concise, efficient, and elegant. This will really deepen your understanding of the material. I encourage you to talk with your colleagues about homework problems, but your final write-up must be your own work.

Homework solutions should be complete (and in particular presented in complete sentences), with all significant steps justified. I reserve the right to

- not grade any homework that is disorganized and incoherent.

Reading for Homework 1

- Sections 1.1 through 1.4 of textbook (this is mostly background material, and can be skimmed)
- Sections 3.1 through 3.3

Problems

4 pts A problem on how the binomial coefficients \( \binom{n}{k} \) change as \( k \) changes.

(a) List, for \( k = 0, \ldots, 8 \), the number of subsets of size \( k \) that a set of size 8 has (give an exact numerical value for each \( k \)).

Solution: 1, 8, 28, 56, 70, 56, 28, 8, 1.
(b) List, for $k = 0, \ldots, 9$, the number of subsets of size $k$ that a set of size 9 has (again, exact numerical values).

**Solution:** 1, 9, 36, 84, 126, 126, 84, 36, 9, 1.

(c) Show that for all even $n$, the number of subsets of size $k$ of a set of size $n$ strictly increases as $k$ goes from 0 to $n/2$, and strictly decreases as $k$ goes from $n/2$ to $n$. [So for even $n$, $n/2$ is the unique most common size of a randomly chosen subset. When $n$ is odd, there are two equally likely most common subset sizes, namely $(n - 1)/2$ and $(n + 1)/2$, but there is no need to prove this].

**Solution:** We know by symmetry that $\binom{n}{k} = \binom{n}{n-k}$, so we only need show that for $1 \leq k \leq n/2$, we have $\binom{n}{k-1} < \binom{n}{k}$. Using $\binom{n}{k} = n!/(k!(n-k)!)$, this is equivalent to

$$\frac{n!}{(k-1)!(n-k+1)!} < \frac{n!}{k!(n-k)!},$$

which by cross-multiplication becomes

$$k < n - k + 1,$$

or $k < (n/2) + 1$, which is indeed true for $1 \leq k \leq n/2$.

4 pts Use the multiplication principle for these three questions:

(a) In how many ways can $n$ male-female couples be seated along a row of $2n$ seats, if each person must sit next to his/her partner?

**Solution:** Choose seat-by-seat, $2n$ choices for first seat, 1 choice for second (partner of person in first), $2n - 2$ choices for third, 1 choice for fourth, etc., leads to

$$2n(2n-2)(2n-4)\ldots(4)(2) = 2^n n!.$$

(b) What if instead it is required that all the women sit in row (but it is not necessary that all the men sit in a row)?

**Solution:** There are $n + 1$ choices for spot of first women in the row of $n$ women (spots 1 through $n + 1$), then $n!$ arrangements of women in those spots, $n!$ arrangements of men in remaining spots, leads t

$$(n + 1)n!n! = (n + 1)!n!.$$

(c) What if instead it is required that every man sits next to his partner, and no two women sit side-by-side?

**Solution:** First think about the gender arrangement along the $2n$ seats. If this arrangement starts MW, it must continue MW MW ... MW to the end. If it starts WM, then second couple can be MW, in which case arrangement must continue MW MW ... MW to end, or second couple can be WM also. In this latter case (starting WMWM) third couple can be MW, in which case arrangement must
continue MW MW ... MW to end, or third couple can be WM. Continuing, we see that arrangement can start with any of 0, 1, 2, ..., n of the couples sitting WM; but as soon as there is the first occurrence of a couple sitting MW, all remaining couples must sit MW. So there are \( n + 1 \) possible gender arrangements. Each one can be filled out in \( n! \) ways: once the location of the men has been determined, the location of the women is forced. Leads to final count of

\[
(n + 1)n! = (n + 1)!.
\]

3 pts Let \( N \) be a set of size \( n \), \( n \geq 1 \). In how many ways can you select a pair of subsets \( A, B \) of \( N \), satisfying \( A \subseteq B \) (\( A \) and \( B \) may be empty). In other words: from among \( n \) people, in how many ways can you form a committee (possibly empty) and a subcommittee from inside that committee (again, possibly empty). Hint: Recast this as the problem of determining the number of functions from \( N \) to \( M \), for some carefully chosen range \( M \).

**Solution:** One way: first decide size \( a \) of committee, \( 0 \leq a \leq n \), then choose committee of that size (\( \binom{n}{a} \) ways), then decide size \( b \) of sub-committee, \( 0 \leq b \leq a \), then choose sub-committee of that size (\( \binom{b}{a} \) ways), leads to ungainly answer

\[
\sum_{a=0}^{n} \binom{n}{a} \sum_{b=0}^{a} \binom{a}{b}.
\]

Another way (the way I was thinking of): Let \( M \) consist of three elements, “not in committee”, “in committee but not in sub-committee”, and “in sub-committee”. A function from \( N \) to \( M \) completely determines the make-up of the committee and the sub-committee, and there are \( 3^n \) such functions, so \( 3^n \) ways to do the selection.

3 pts Use induction to verify this identity that we conjectured from looking at Pascal’s triangle: Fix \( n \geq 0 \) and \( k \geq 0 \).

\[
\sum_{\ell=0}^{k} \binom{n + \ell}{\ell} = \binom{n}{0} + \binom{n + 1}{1} + \ldots + \binom{n + k}{k} = \binom{n + k + 1}{k}.
\]

**Solution:** Induction on \( n \) seems to go nowhere, so try induction on \( k \). Statement \( P(k) \) is “for \( n \geq 0 \),

\[
\binom{n}{0} + \binom{n + 1}{1} + \ldots + \binom{n + k}{k} = \binom{n + k + 1}{k}''
\]

Base case, \( k = 0 \): \( P(0) \) is statement “for \( n \geq 0 \), \( \binom{n + 0}{0} = \binom{n + 0 + 1}{0}'' \), which is true since it reduces to \( 1 = 1 \).

Inductive step: Assume \( P(k) \) is true for some \( k \geq 0 \). I.e., assume that for all \( n \geq 0 \),

\[
\binom{n}{0} + \binom{n + 1}{1} + \ldots + \binom{n + k}{k} = \binom{n + k + 1}{k}.
\]
Adding \( \binom{n+k+1}{k+1} \) to both sides, this becomes
\[
\binom{n}{0} + \binom{n+1}{1} + \ldots + \binom{n+k}{k} + \binom{n+k+1}{k+1} = \binom{n+k+1}{k} + \binom{n+k+1}{k+1}.
\]

Applying Pascal’s identity to right-hand side above, it becomes
\[
\binom{n}{0} + \binom{n+1}{1} + \ldots + \binom{n+k}{k} + \binom{n+k+1}{k+1} = \binom{n+k+2}{k+1}
\]
or
\[
\binom{n}{0} + \binom{n+1}{1} + \ldots + \binom{n+k}{k} + \binom{n+k+1}{k+1} = \binom{n+(k+1)+1}{k+1},
\]
which is \( P(k+1) \).

The truth of \( P(k) \) for all \( k \geq 0 \) follows by induction.

3 pts Here’s an identity that we proved in class by induction: Fix \( n \geq k \geq 0 \).
\[
\sum_{\ell=k}^{n} \binom{\ell}{k} = \binom{k+1}{k} + \binom{k+1}{k} + \ldots + \binom{n}{k} = \binom{n+1}{k+1}.
\]

Give a combinatorial proof of this. More specifically, the right-hand side of the identity counts the number of subsets of size \( k+1 \) of a set of size \( n+1 \). Show that the left-hand side can be interpreted as counting the same thing.

**Solution:** Let the set of size \( n+1 \) be \( \{1, \ldots, n+1\} \). A subset of size \( k+1 \) of this set has a largest element, which is one of \( k+1, k+2, \ldots, n+1 \). There are \( \binom{k}{k} \) subsets of size \( k+1 \) with largest element \( k+1 \) (the remaining \( k \) elements have to be chosen from \( \{1, \ldots, k\} \)); there are \( \binom{k+1}{k} \) subsets of size \( k+1 \) with largest element \( k+2 \) (the remaining \( k \) elements have to be chosen from \( \{1, \ldots, k+1\} \)); and so on, up to: there are \( \binom{n}{k} \) subsets of size \( k+1 \) with largest element \( n+1 \) (the remaining \( k \) elements have to be chosen from \( \{1, \ldots, n\} \)). It follows that the number of such subsets is
\[
\sum_{\ell=k}^{n} \binom{\ell}{k} = \binom{k+1}{k} + \binom{k+1}{k} + \ldots + \binom{n}{k}.
\]

3 pts Exercise 6 from Section 1.3 of the textbook (page 23)

**Solution:** We proceed by induction on \( n \), with the base case \( n = 0 \) (1 by 1 board) trivial. For \( n > 0 \), divide the \( 2^n \) by \( 2^n \) board into four \( 2^{n-1} \) by \( 2^{n-1} \) subboards by putting vertical and horizontal lines down and across the middle of the board. One of these subboards contains the special square, and by induction that subboard can be covered so that only the special square is left uncovered. Also by induction, each of the other three subboard can be covered so that only the square of the subboard closest to the center of the board is left uncovered. These three uncovered squares can be covered with a single L-shaped tile.
4 A combinatorial proof of the binomial theorem

The binomial theorem says that for all real (or complex) $x$ and $y$, and for all integers $n \geq 0$, we have

$$(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$\

To make sense of some silly boundary instances, we interpret $0^0$ as 1 whenever it occurs.

In class we gave a combinatorial proof when $x$ and $y$ are positive integers. Here we explain how this combinatorial proof can be used to get the result for all $x, y$.

We begin by noting that when $n = 0$ the result is clear — by our convention on $0^0$, the identity quickly reduces to $1 = 1$ for all possible choices of $x$ and $y$). So from now on we assume $n \geq 1$.

If $y = 0$ then (again by our convention on $0^0$) the identity easily reduces to $x^n = x^n$ for all possible choices of $x$, so it is trivially true. So from now on we assume $y \neq 0$.

Dividing by $y^n$, the left-hand side becomes

$$\frac{(x + y)^n}{y^n} = \left(\frac{x}{y} + 1\right)^n$$

and the right-hand side becomes

$$\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} = \sum_{k=0}^{n} \binom{n}{k} \left(\frac{x}{y}\right)^k.$$

Setting $z = x/y$, we see that what we have left to proof is

$$(z + 1)^n = \sum_{k=0}^{n} \binom{n}{k} z^k$$

for all $z$ and all $n \geq 1$.

Now comes the combinatorics. If $z$ is a positive integer, then the left-hand side above counts the number of words of length $n$ from alphabet $\{0, 1, 2, \ldots, z\}$, by constructing words one letter at at time. The right-hand side counts the same thing, by first choosing $k$, the number of letters that comes from $\{1, 2, \ldots, z\}$ (choices for $k$ are 0 through $n$), then choosing the exact location of those $k$ letters in the word ($\binom{n}{k}$ options), then choosing the actual letters that go into those locations, one after another ($z$ options for the first location, $z$ for the second, etc., so $z^k$ for all $k$). This completely determines the word, since all remaining locations get letter 0.

This combinatorial argument shows that $(z + 1)^n = \sum_{k=0}^{n} \binom{n}{k} z^k$ holds for $z = 1, 2, 3, \ldots$.

How do we get the equality for other values of $z$? Notice that $(z + 1)^n$ is a polynomial of degree $n$, and $\sum_{k=0}^{n} \binom{n}{k} z^k$ is also a polynomial of degree $n$. We want to say that they are the same polynomial. Could they be different? Well, two different polynomials of degree $n$ can only agree at $n$ values. If they agree at $n + 1$ or more values, then their difference is a non-zero polynomial of degree at most $n$ that has at least $n + 1$ roots, a contradiction! (A non-zero polynomial of degree $n$ can have at most $n$ roots).
But our two polynomials — \((z + 1)^n\) and \(\sum_{k=0}^{n} \binom{n}{k} z^k\) — agree at infinitely many values, and so certainly agree at least \(n+1\). We conclude that they have to be the same polynomial, and so indeed

\[(z + 1)^n = \sum_{k=0}^{n} \binom{n}{k} z^k\]

for all \(z\) and all \(n \geq 1\).

This gives a more-or-less completely combinatorial proof of the binomial theorem. The trick of proving an identity involving a single variable by

1. first proving the identity combinatorially for all positive integer inputs,
2. then observing that both sides of the identity are polynomials
3. then concluding (for the reason described above) that the polynomials must agree, so the identity must be true for all inputs

is called the polynomial principle.

There is a slightly fancier polynomial principle, that’s easy enough to prove, though I won’t give the proof here: if \(f_1(x_1, \ldots, x_k)\) and \(f_2(x_1, \ldots, x_k)\) are multi-variable polynomials, and they agree whenever all the inputs \(x_i\) are positive integers, then they are identical.

Using the fancier polynomial principle, the combinatorial proof of the binomial theorem is even quicker, as it doesn’t require any of the preparatory work we did to reduce the problem to one variable. We simply say that

\[(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}\]

holds for all positive integers \(x\) and \(y\), using the counting proof we gave in class, and then we are instantly done; the polynomial principle gives the identity for all \(x, y\)!
5 Homework 2 — due Thursday September 10

Reading for Homework 2

• Section 3.3

Problems

4 pts (a) In how many ways can \( k \) indistinguishable tokens be distributed among \( n \) distinguishable people, if it is not required that all the tokens be given out (and it is not required that everyone get at least one token)? Your answer should not involve a sum of terms. **Hint:** Create a phantom \((n + 1)\)st person.

**Solution:** Create a phantom \((n + 1)\)st person who receives all the undistributed tokens. We are now faced with the problem of distributing \( k \) tokens among \( n + 1 \) people; by results from class, this can be done in

\[
\binom{k + (n + 1) - 1}{(n + 1) - 1} = \binom{k + n}{n}
\]

ways.

(b) In how many ways can \( k \) indistinguishable tokens be distributed among \( n \) distinguishable people, if it is not required that all the tokens be given out, but it is required that everyone get at least two tokens? Your answer should not involve a sum of terms.

**Solution:** First distribute 2 tokens to each person; now we are left with exactly the problem from part a), with \( “k” \) replace by \( “k – 2n” \), leading to a count of

\[
\binom{(k – 2n) + n}{n} = \binom{k – n}{n}.
\]

Notice that this evaluates to 0 unless \( k \geq 2n \).

3 pts Pick any entry in Pascal’s triangle, other than one of the 1’s on the boundary (see e.g. [http://ptri1.tripod.com/](http://ptri1.tripod.com/) for a picture of Pascal’s triangle). There are six numbers on the triangle that are arranged around it in a circle (e.g., if you had picked the “3” below that is the third entry of row 3, then the six numbers, read in a clockwise circular order, would be 2,1,1,4,6,3). Prove that the product of the first, third and fifth numbers (2,1,6 in our example, product 12) is equal to the product of the second, fourth, sixth (1,4,3 in the example, product also 12). **Hint:** Let the number you select be \( \binom{n}{k} \). Formulate the claim to be proved as an identity involving products of binomial coefficients. Use the algebraic definition of binomial coefficients, rather than the combinatorial one.
Solution: The identity is

\[
\binom{n-1}{k-1} \binom{n}{k+1} \binom{n+1}{k} = \binom{n-1}{k} \binom{n}{k-1} \binom{n+1}{k+1},
\]

which is easily verified using \( \binom{a}{b} = \frac{a!}{b!(a-b)!} \). (Everything on both sides cancels.)

3 pts Use induction on \( n \) to prove Leibniz’ formula for the iterated product rule: if \( f \) and \( g \) are functions of a single variable, and \( (n) \) denotes the \( n \)th derivative with respect to that variable, then for all \( n \geq 1 \),

\[
(fg)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(k)} g^{(n-k)}.
\]

Here the zeroth derivative of a function is just the function itself. You can assume the usual product rule \((fg)' = fg' + f'g\).

Solution: Base case \( n = 1 \):

\[
(fg)^{(1)} = fg' + f'g = \binom{1}{0} f^{(0)} g^{(1)} + \binom{1}{1} f^{(1)} g^{(0)} = \sum_{k=0}^{1} \binom{1}{k} f^{(k)} g^{(1-k)},
\]

as required.

Induction step: Assuming

\[
(fg)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(k)} g^{(n-k)} = \binom{n}{0} f^{(0)} g^{(n)} + \ldots + \binom{n}{k} f^{(k)} g^{(n-k)} + \ldots + \binom{n}{n} f^{(n)} g^{(0)},
\]

Rest later...

5 pts Fix \( n \geq 1 \). Let \( S_n^{\text{even}} \) be the set of even-sized subsets of a set of size \( n \) (say, for concreteness, the set \( \{1, 2, \ldots, n\} \)). Let \( S_n^{\text{odd}} \) be the set of odd-sized subsets. Write down a bijection \( f : S_n^{\text{even}} \rightarrow S_n^{\text{odd}} \) (by this I mean, give the value of \( f(A) \) for each \( A \in S_n^{\text{even}} \)). Verify that the function you have written down does indeed have \( S_n^{\text{odd}} \) as its range (i.e., that \( f(A) \in S_n^{\text{odd}} \) for all \( A \in S_n^{\text{even}} \)). Verify also that \( f \) is both injective and surjective. (This gives a combinatorial proof of the identity

\[
\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \ldots = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \ldots
\]

Solution: Define \( f : S_n^{\text{even}} \rightarrow S_n^{\text{odd}} \) by

\[
f(A) = \begin{cases} 
A \cup \{1\} & \text{if } 1 \not\in A \\
A \setminus \{1\} & \text{if } 1 \in A.
\end{cases}
\]

First note that \( f(A) \) is a subset \( \{1, \ldots, n\} \) of size \( |A| \pm 1 \), which is odd if \( |A| \) is even, so \( f \) is indeed a function from \( S_n^{\text{even}} \) to \( S_n^{\text{odd}} \).
To check injectivity, consider $A \neq B \in S_{n}^{\text{even}}$. Since $A \neq B$, there must be some $a$ such that either $a \in A, a \notin B$ or $a \notin A, a \in B$. Without loss of generality we assume that there is some $a$ with $a \in A, a \notin B$. If $a = 1$ then $a \notin f(A), a \in f(B)$ so $f(A) \neq f(B)$. If $a \neq 1$ then $a \in f(A), a \notin f(B)$ so also $f(A) \neq f(B)$. Either way, $f(A) \neq f(B)$ so $f$ is injective.

To check surjectivity, consider $B \in S_{n}^{\text{odd}}$. If $1 \in B$ then $A := B \setminus \{1\}$ is in $S_{n}^{\text{even}}$ and satisfies $f(A) = B$. If $1 \notin B$ then $A := B \cup \{1\}$ is in $S_{n}^{\text{even}}$ and satisfies $f(A) = B$. So $f$ is indeed surjective.

3 pts  (a) What is the coefficient of $x^2y^3zw^4$ in the polynomial $(2x - y + 3z + w)^{10}$?

**Solution:** From the multinomial theorem the answer is

$$\binom{10}{2, 3, 1, 4} \times (-1)^3 \times 3 \times 1 = -151200.$$ 

(b) The polynomial $(x + 2y + 3z)^8$ can be viewed as a polynomial in $x$ with the coefficients depending on $y$ and $z$. What is the coefficient of $x^3$ in this viewpoint?

**Solution:** Set $w = 2y + 3z$. The coefficient of $x^3$ in $(x + 2y + 3z)^8$ is the same as the coefficient of $x^3$ in $(x + w)^8$. By the binomial theorem this is

$$\binom{8}{3} w^5 = 56 (2y + 3z)^5.$$ 

1. (Not for credit, but for a small prize — I’ll put all correct answers into a hat, and draw one at random to get a prize). Fix $n \geq 1$. Show that

$$\binom{n}{0} + \binom{n}{3} + \binom{n}{6} + \ldots$$

is within $\pm 1$ of $2^n/3$. [In other words: the sum of every third entry of any row of Pascal’s triangle is always within 1 of one third of the total row sum].

**Solution:** Later.
6 Homework 3 — due Thursday September 17

Reading for Homework 3

• Sections 3.3, 3.7 and 3.8

Problems

3 pts How many numbers are left in the set \{1, \ldots, 1000\} after all the multiples of 2, 3, 5 and 7 are crossed out? [This is start of the sieve of Eratosthenes, used around 200BC to find all the prime numbers less than a given number; see https://en.wikipedia.org/wiki/Sieve_of_Eratosthenes for a lovely graphic illustration of the full method.]

Solution: Let \( A_i \) be the number of numbers in \{1, \ldots, 1000\} that are multiples of \( i \). We have
\[
|A_2| = 500, \quad |A_3| = 333, \quad |A_5| = 200, \quad |A_7| = 142,
\]
\[
|A_2 \cap A_3| = 166, \quad |A_2 \cap A_5| = 100, \quad |A_2 \cap A_7| = 71, \quad |A_3 \cap A_5| = 66, \quad |A_3 \cap A_7| = 47, \quad |A_5 \cap A_7| = 28,
\]
\[
|A_2 \cap A_3 \cap A_5| = 33, \quad |A_2 \cap A_3 \cap A_7| = 23, \quad |A_2 \cap A_5 \cap A_7| = 14, \quad |A_3 \cap A_5 \cap A_7| = 9
\]
and
\[
|A_2 \cap A_3 \cap A_5 \cap A_7| = 4.
\]
(To calculate \( |A_2 \cap A_3 \cap A_5| \) note that being in \( A_2 \cap A_3 \cap A_5 \) is equivalent to being divisible by all of 2, 3 and 5, which is equivalent to being divisible by \( 2 \cdot 3 \cdot 5 = 30 \), so \( A_2 \cap A_3 \cap A_5 \) is the set of multiples of 30 at or below 1000; since \( 1000/30 = 33.33 \ldots \), there are 33 such multiples; the rest of the sets are calculated in the same way).

By inclusion-exclusion, \( |A_2 \cup A_3 \cup A_5 \cup A_7| \) is
\[
(500 + 333 + 200 + 142) - (166 + 100 + 71 + 66 + 47 + 28) + (33 + 23 + 14 + 9) - 4 = 772.
\]
So there are 772 multiples of 2, 3, 5 and 7 that get crossed out, leaving 228 numbers.

3 pts (a) In how many ways can 4 Russians, 4 Americans and 4 Canadians be lined up in row, so that no nationality forms a single consecutive block?

Solution: Plug in \( n = 4 \) to part (b) to get 454 035 456 ways.

(b) In how many ways can \( n \) Russians, \( n \) Americans and \( n \) Canadians be lined up in row, so that no nationality forms a single consecutive block? [Here it might be helpful to use the identity \( 1 + 2 + 3 + \ldots + k = k(k+1)/2 \), which can be easily proven by induction.]

Solution: Let \( U \) be total number of arrangements, let \( A_R \) be those in which four Russians are sitting side-by-side, and define \( A_A \) and \( A_C \) similarly. We want
\[
|U| - |A_R \cup A_A \cup A_C|.
\]
We have \( |U| = (3n)! \), and \( |A_R| = |A_A| = |A_C| = n!(2n+1)! \). To see this, note that there are \( n! \) ways of ordering the Russians, and once we are committed to putting them together in a block, we are in essence dealing with a problem of arbitrarily arranging \( 2n+1 \) people: \( n \) Americans, \( n \) Canadians and one "super-Russian" (the ordering of the \( n \) Russians, who must stay together in a block).
By similar reasoning, $|A_R \cap A_A| = |A_R \cap A_C| = |A_A \cap A_C| = (n!)^2(n + 2)!$ and $|A_R \cap A_A \cap A_C| = (n!)^33!$. 

By inclusion-exclusion,

$$|A_R \cup A_A \cup A_C| = 3n!(2n + 1)! - 3(n!)^2(n + 2)! + 6(n!)^3,$$

and so

$$|U| - |A_R \cup A_A \cup A_C| = (3n)! - 3n!(2n + 1)! + 3(n!)^2(n + 2)! - 6(n!)^3.$$

3 pts $n$ male-female couples attend a dance. In how many ways can they pair off into $n$ male-female pairs to dance, if no pair of partners should dance with one another? [E.g., if $n = 3$, with the couples $Aa, Bb, Cc$, then they could pair off as $Ab, Bc, Ca$ or $Ac, Ba, Cb$, so there are two ways. Your answer will involve a summation.]

**Solution:** Label the $n$ couples 1 through $n$. A way to pair off the couples to dance is to generate a permutation of $\{1, \ldots, n\}$, and if the permutation sends $i$ to $j$, then have the man in couple $i$ dance with the woman in couple $j$. To avoid any pair of partners dancing with one another, this permutation must be a derangement. So the answer is the number of derangements of $\{1, \ldots, n\}$, or

$$d_n = \sum_{k=0}^{n} (-1)^k \frac{n!}{k!}.$$

3 pts Remember that $d_n$ counts the number of derangements of $n$ objects. By convention we set $d_0 = 1$. Show that for any $n \geq 0$ we have the identity

$$n! = \sum_{k=0}^{n} \binom{n}{k} d_k.$$

[This gives a recursive way to calculate $d_n$ in terms of $d_{n-1}, d_{n-2}, \ldots, d_0$.]

**Solution:** The left-hand side counts the number of permutations of $\{1, \ldots, n\}$, deciding the location of the elements one-by-one.

The right-hand sides also counts the number of permutations of $\{1, \ldots, n\}$, by the following method. First, decide on $k$, the number of elements that are *not* sent to themselves in the permutation; $k$ varies between 0 and $n$. Next, decide which $k$ elements are not sent to themselves; there are $\binom{n}{k}$ options here. Next, derange those $k$ elements among themselves (to ensure that none is sent to itself); there are $d_k$ options here. Finally, send all other elements to themselves (one option). For each $k$ there are $\binom{n}{k} d_k$ permutations that have elements that are not sent to themselves, so there are $\sum_{k=0}^{n} \binom{n}{k} d_k$ permutations in total.

Since both sides count the same thing, they are equal.
One last binomial coefficient identity: give a combinatorial proof that

\[ \binom{r + s}{t} = \sum_{k=0}^{t} \binom{r}{k} \binom{s}{t-k}. \]

**Solution:** Consider selecting a committee of size \( t \) from among \( r \) men and \( s \) women.

The gender-blind selection process gives a total of \( \binom{r+s}{t} \) committees.

There is also a gender-conscious selection process. First decide on the value or \( k \), the number of men on the committee; \( k \) varies between 0 and \( t \). Then select a committee of \( k \) men \( (\binom{r}{k} \text{ options}) \) and \( t - k \) women \( (\binom{s}{t-k} \text{ options}) \). This process gives a total of \( \sum_{k=0}^{t} \binom{r}{k} \binom{s}{t-k} \) committees.

Since both sides count the same thing, they are equal.
7 Homework 4 — due Thursday September 24

Reading for Homework 4

• Sections 3.7, 3.8 and 4.1
• Lecture notes on Stirling numbers of the second kind

Problems

1. 1 pt For \( n \geq 1 \), argue that
\[
|A_1 \cup A_2 \cup \ldots \cup A_n| \leq |A_1| + |A_2| + \ldots + |A_n|
\]

Solution: Anything outside \( A_1 \cup A_2 \cup \ldots \cup A_n \) is counted zero times on both sides. Anything in \( A_1 \cup A_2 \cup \ldots \cup A_n \) is counted exactly once on the left-hand side. On the right-hand side, it is counted exactly \( k \) times if it is in exactly \( k \) of the \( A_i \)'s. Since \( k \geq 1 \), it is counted at least once on the right-hand side. Hence, right-hand size is at least as large as the left-hand side.

3 pts For \( n \geq 2 \), argue that
\[
|A_1 \cup \ldots \cup A_n| \geq |A_1| + \ldots + |A_n| - |A_1 \cap A_2| - \ldots - |A_{n-1} \cap A_n|.
\]

(the right-hand side includes all \( \binom{n}{2} \) pair-intersections from among \( A_1, \ldots, A_n \)).

Solution: Anything outside \( A_1 \cup A_2 \cup \ldots \cup A_n \) is counted zero times on both sides. Anything in \( A_1 \cup A_2 \cup \ldots \cup A_n \) is counted exactly once on the left-hand side, so it enough to show that such an element contributes at most +1 to the right-hand side. Suppose the element of \( A_1 \cup A_2 \cup \ldots \cup A_n \) under consideration appears in exactly \( k \) of the \( A_i \)'s (\( k \geq 1 \)), say, without loss of generality, \( A_1, \ldots, A_k \). If \( k = 1 \), this element contributes exactly +1 to the right-hand side (from \( |A_1| \)). If \( k = 2 \), it contribute +1 (from \( |A_1|, |A_2|, -|A_1 \cap A_2| \)). For \( k \geq 3 \), it contributes
\[
k - \binom{k}{2} = \frac{-k^2 + 3k}{2}
\]

(from \( |A_1| \) through \( |A_k| \) and \(-|A_1 \cap A_2| \) through \(-|A_{k-1} \cap A_k|\)). For \( k \geq 3 \), \( k^2 \geq 3k \) so \((-k^2 + 3k)/2 \leq 0 \), as required.

2 pts In class we saw the Stirling number of the second kind, \( \left\{ \begin{array}{c} n \\ k \end{array} \right\} \), that counts the number of ways that a set of size \( n \) can be partitioned into \( k \) non-empty blocks, if neither order of the blocks nor order within the blocks matters.

Find a simple expression for \( \left\{ \begin{array}{c} n \\ n-1 \end{array} \right\} \) (your answer should a polynomial in \( n \)).

Solution: To partition set of size \( n \) into \( n-1 \) non-empty blocks, it is necessary to have one block of size 2 and the rest of size 1. The only choice we have is in the two elements that go into the block of size 2, for which there are \( \binom{n}{2} \) options. Hence
\[
\left\{ \begin{array}{c} n \\ n-1 \end{array} \right\} = \binom{n}{2} = \frac{n^2}{2} - \frac{n}{2}.
\]
2. The $n$th Bell number $B_n$ is defined to be the number of ways that a set of size $n$ can be partitioned into (any number of) non-empty blocks. [So $B_n$ is the sum of the entries of the $n$th row of the Stirling equivalent of Pascal’s triangle].

3 pts Show that the Bell numbers can be calculated using the recurrence relation

$$B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} B_k$$

for $n \geq 0$, with initial condition $B_0 = 1$.

**Solution**: Since we defined $\binom{0}{0} = 1$, and since $\binom{0}{k} = 0$ for $k \geq 0$, we get $B_0 = 1$. For the recurrence, we give a combinatorial proof. Let $S_{n+1}$ be the set of all partitions of $\{1, \ldots, n + 1\}$ into non-empty blocks, and, for $k = 0, \ldots, n$ let $S_{n+1}^k$ be the set of all partitions of $\{1, \ldots, n + 1\}$ into non-empty blocks in which there are exactly $k$ elements of $\{1, \ldots, n\}$ that are not in the same block as element $n + 1$. We have

$$S_{n+1} = \bigcup_{i=0}^{n} S_{n+1}^i,$$

and the $S_{n+1}^i$’s are disjoint for different $k$s. Hence

$$|S_{n+1}| = \sum_{i=0}^{n} |S_{n+1}^i|.$$

Evidently $|S_{n+1}| = B_{n+1}$, and $|S_{n+1}^k| = \binom{n}{k} B_k$ since first we must choose the $k$ elements from $\{1, \ldots, n\}$ that are not in the same block as element $n + 1$ ($\binom{n}{k}$ choices), then we must choose a partition of those $k$ elements into non-empty blocks ($B_k$ choices) (note that we don’t need to determine the block of $n + 1$ since that is fully determined from the choice of elements not in the same block as $n + 1$). The result follows.

1 pt *Use the recurrence from part (a)* to verify that $B_4 = 15$.

**Solution**: Initial condition: $B_0 = 1$. Recurrence when $n = 0$:

$$B_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} B_0 = 1.$$

Recurrence when $n = 1$:

$$B_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} B_0 + \begin{pmatrix} 1 \\ 1 \end{pmatrix} B_1 = 2.$$

Recurrence when $n = 2$:

$$B_3 = \begin{pmatrix} 2 \\ 0 \end{pmatrix} B_0 + \begin{pmatrix} 2 \\ 1 \end{pmatrix} B_1 + \begin{pmatrix} 2 \\ 2 \end{pmatrix} B_2 = 5.$$

Recurrence when $n = 3$:

$$B_4 = \begin{pmatrix} 3 \\ 0 \end{pmatrix} B_0 + \begin{pmatrix} 3 \\ 1 \end{pmatrix} B_1 + \begin{pmatrix} 3 \\ 2 \end{pmatrix} B_2 + \begin{pmatrix} 3 \\ 3 \end{pmatrix} B_3 = 15.$$
.75 pts per part Section 4.1, problem 2 (page 117 of the textbook). For each of the eight parts, say TRUE or FALSE below, and give a (brief, please!) justification of your choice.

(i) FALSE — As written, the statement is nonsense: the only way that \( \{f(u), f(v)\} \in V(H) \) can happen is if \( f(u) = f(v) \), since \( V(H) \) is a set of vertices, not pairs of vertices. If you scanned and assumed (as I did) that the statement ended “\( f(u) = f(v) \in E(H) \)”, then it’s still a false statement. The implication needs only be true for some bijection, not all functions.

(ii) FALSE — right-hand side of implication just says that \( G \) and \( H \) have the same number of edges, which is not in general sufficient for isomorphism, as we saw in class with examples of two non-isomorphic graphs with three edges on four vertices.

(iii) FALSE — right-hand side of implication is necessary but not in general sufficient for isomorphism, as we saw in class with a cycle on six vertices versus two side-by-side cycles on three vertices each.

(iv) TRUE — any bijection that satisfies the defining property of an isomorphism preserves vertex degree as it moves from \( G \) to \( H \).

(v) TRUE — If \( G \) and \( H \) are isomorphic they must have the same numbers of edges, so there are bijections between the edge-sets.

(vi) FALSE — such a map must be a bijection. If \( G \) has vertices \( \{1, \ldots, 5\} \) and \( H \) has vertices \( \{1, \ldots, 4\} \), and neither have any edges, then they are not isomorphic, but the map from \( V(G) \) to \( V(H) \) that sends everything to 1 satisfies the conditions given.

(vii) TRUE — If the vertex set of \( G \) is \( \{a_1, \ldots, a_n\} \), and if \( H \) is a graph on vertex set \( \{1, \ldots, n\} \) with an edge from \( i \) to \( j \) if and only if \( a_i \) and \( a_j \) are joined in \( G \), then the map from \( V(G) \) to \( V(H) \) that sends \( a_i \) to \( i \) for each \( i \) is an isomorphism.

(viii) TRUE — there are infinitely many names we could change the names of the vertices to (e.g., the graph on one vertex named “1” is isomorphic to the graph on one vertex named “n” for all integers \( n \).

3 pts Section 4.1, problem 3, part (a) (page 117 of the textbook)

**Solution:** Lot of examples. Here’s one: vertices \( a, b, c, d, e, f, g \), edges \( ab, ac, ce, ad, df, fg \). In any automorphism:

- \( a \) must map to \( a \), as it is the only vertex of degree 3
- \( b \) must map to \( b \), as it is the only vertex of degree 1 with a neighbor of degree 3
- \( c \) must map to \( c \), as it is the only vertex of degree 2 with neighbors of degree 1 and 3
- \( d \) must map to \( d \), as it is the only vertex of degree 2 with neighbors of degree 2 and 3
- \( e \) must map to \( e \), as it must map to a neighbor of \( c \), and \( e \) is the only one left
- \( f \) must map to \( f \), as it must map to a neighbor of \( d \), and \( f \) is the only one left
- \( g \) must map to \( g \), as this is the only vertex left.
So the only automorphism is the identity, as required.
8  Homework 5— due Thursday October 1

Reading for Homework 5

• Sections 4.1, 4.2, 4.3

Problems

3 pts Section 4.2, Exercise 1 (page 123 of textbook)

Solution: Let x and y be two different vertices in G. If x and y are in different components of G, then there is no edge joining them, so in the complement of G there is an edge joining them. If x and y are in the same component, then pick an arbitrary vertex z that is not in the same component as x or y (there must be some such z since G is not connected). In the complement of G there is a edge from x to z and one from z to y. It follows that the complement of G is connected [and moreover that between any two vertices in the complement of G, there is a path of length at most 2].

4 pts Section 4.2, Exercise 10 (page 124)

Solution: For one direction, suppose x, y and z form the vertices of a triangle in G. Then the $xy$ entry of $A_G$ is 1, and, since there is at least one walk of length 2 from x to y (the walk $(x, xz, z, zy, y)$), the $xy$ entry of $A_G^2$ is at least 1 (remember that the $xy$ entry of $A_G^2$ counts the number of walks of length 2 from x to y).

For the other direction, suppose that both the $xy$ entry of $A_G$ and $A_G^2$ is non-zero. It follows that $xy$ is an edge of G, and that there is at least one walk of length 2 from x to y. Let $(x, xz, z, zy, y)$ be one such; note that $z \neq x, z \neq y$ since graphs don’t have loops. But then x, y and z form the vertices of a triangle in G.

(a) Show that if there is a walk from x to y of length t, then there is a path from x to y of length at most t.

Solution: Let $(v_0, e_1, v_1, e_2, v_2, \ldots, v_{t-1}, e_t, v_t)$, with $x = v_0, y = v_t$ be a walk of length t from x to y. If this walk is a path, we are done. If it is not a path, then there are $v_a$ and $v_b$, $0 \leq a < b \leq t$, with $v_a = v_b$. Consider the string

$$(v_0, e_1, v_1, e_2, v_2, \ldots, v_a, e_{b+1}, \ldots, v_{t-1}, e_t, v_t)$$

obtained from the walk by “short-circuiting” the part of the path between $v_a$ and $v_b$. The result is a shorter walk from x to y (note that it is still a legitimate, non-empty walk: we cannot have $a = 0, b = t$, since that would imply $x = y$, and it seems implicit in the question that $x \neq y$).
If this shorter walk is a path, we are done; if not, repeated the process of short-
circuiting. Since we can never get to an empty string, the process must terminate 
at some point in a walk from \textit{x} to \textit{y} that has no repeated vertices, that is in a 
path from \textit{x} to \textit{y}, that has length at most \textit{t}.

(b) Deduce that the distance function on a graph satisfies the triangle inequality: 
\[ d_G(x, y) \leq d_G(x, z) + d_G(z, y). \]

\textbf{Solution:} The equality is trivial when \textit{x} = \textit{y}, so assume \textit{x} \neq \textit{y}. If \[ d_G(x, z) = m \]
and \[ d_G(z, y) = n, \] then there is an \( x - z \) path of length \( m \) and a \( z - y \) path 
of length \( n \) in \( G \). Concatenating these gives an \( x - y \) walk (not necessarily a 
path — the \( x - z \) part may intersect with the \( z - y \) part) of length \( m + n \). By 
part a), there is an \( x - y \) path of length at most \( m + n \), so \( d_G(x, y) \leq m + n = 
d_G(x, z) + d_G(z, y). \)

3 pts Section 4.3, Exercise 5 (page 129)

\textbf{Solution:} There are only seven vertices, so the vertex of degree six must be adjacent 
to all of the other six. Just looking at the subgraph induced by those other six, we 
have a graph with score \( (2, 2, 2, 2, 2, 2, 2) \). Say, without loss of generality, that the vertices 
are \( x_1, \ldots, x_6 \) and that \( x_2 \) is adjacent to \( x_1 \) and \( x_3 \). If \( x_3 \) is adjacent \( x_1 \), then \( x_1, x_2, x_3 \) 
all have degree 2, and the other way to complete the graph is to join \( x_4, x_5 \) and 
\( x_6 \) in a triangle. If \( x_3 \) is not adjacent to \( x_1 \), it must be adjacent to one of \( x_4, x_5 \) and 
\( x_6 \), without loss of generality \( x_4 \). If \( x_1 \) is also adjacent to \( x_4 \), then \( x_1, x_2, x_3, x_4 \) now 
all have degree 2, but there is no way to get \( x_5, x_6 \) having degree 2 each. So assume, 
without loss of generality, that \( x_1 \) is adjacent to \( x_6 \). Now \( x_1, x_2, x_3, x_4 \) all have degree 
2, and \( x_5, x_6 \) have degree 1 each, so the only way to complete is to join \( x_5 \) and \( x_6 \) to 
create a six-cycle.

Conclusion: There are two graphs with score \( (6, 3, 3, 3, 3, 3, 3) \) — a six-cycle with a 
seventh vertex adjacent to everything, and two disjoint triangles with a seventh vertex 
as adjacent to everything.

3 pts Section 4.3, Exercise 9 (page 129)

\textbf{Solution:} Build the path greedily. Pick an arbitrary vertex \textit{v}_0. It has \textit{d} neighbours, 
pick one arbitrarily, say \textit{v}_1. \textit{v}_1 \ has \textit{d} neighbours, at most 1 of which is \textit{v}_0, so it has 
at least \( d - 1 \) neighbours that haven’t yet been selected, pick one of them, \textit{v}_2 \ say. \textit{v}_2 
has \textit{d} neighbours, at most 2 of which are \textit{v}_0, \textit{v}_1, so it has at least \( d - 2 \) neighbours that 
haven’t yet been selected, pick one of them, \textit{v}_3 \ say. Repeat. When we reach \textit{v}_k, \textit{v}_k \ has 
d neighbours, at most \( k \) of which are \textit{v}_0, \textit{v}_1, \ldots, \textit{v}_{k-1}, \) so it has at least \( d - k \) neighbours 
that haven’t yet been selected, pick one of them, \textit{v}_{k+1} say.

This works as long as \( d - k > 0 \), i.e., as long as \( k \leq d - 1 \). Thus we are certain to get 
a path \((\textit{v}_0, \textit{v}_0\textit{v}_1, \textit{v}_1\textit{v}_2, \textit{v}_2, \ldots, \textit{v}_{d-1}\textit{v}_d, \textit{v}_d)\) of length \textit{d}.

[Note that there is no certainty of a path of length \textit{d} + 1: \( K_{d+1} \) is \textit{d}-regular and has 
longest path length \textit{d}.]
Simpler solution: Let $P$ be a longest path in $G$. Let $v$ be one of the end-vertices of $P$. Because $P$ is a longest path, $v$ can have no neighbors that are not on the path (else the path could be extended). Since $v$ has at least $d$ neighbors, $P$ has at least $d + 1$ vertices ($v$ and its at least $d$ neighbors), so has length at least $d$. 
9 The parallel mountain climbers problem

A linear mountain range starts at $A$, ends at $Z$ and has a highest point $M$ somewhere in the middle. Is it possible for two climbers, one starting at $A$, one starting at $Z$, to move along the mountain range and meet at $M$, while all the time staying at the same elevation as each other? In other words, whenever one climber is going up, the other must be too, and whenever one is going down, the other must be too; and whatever height above sea-level one of them is at at any given moment, the other must be at the same height.

Formally we can think of a mountain range as a sequence of points in the plane $A = (x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n) = Z$, with $x_1 < x_2 < \ldots < x_n$ and with the $y_i$’s alternating: $y_1 < y_2 > y_3 < y_4 > \ldots$ (so the $(x_i, y_i)$’s are alternating peaks and valleys). But really we will just think about mountain ranges informally, as in the picture below.

There are two conditions that a mountain range must satisfy, for there to be any chance for the climbers to succeed:

1. $A$ and $Z$ must be at the same level, call it sea-level (so the climbers can start), and

2. no valley dips below sea level (if it did, say on the left side of $M$, then the left climber would be unable to cross this valley unless he was lucky enough that there was a corresponding dip on the right side).

With them both, for fairly simple mountain ranges the problem is easy to solve by hand. But as they get more complex, it gets less and less clear that it is always possible for the mountain climbers to succeed. We’ll show that they always can, using the handshake lemma.

**Theorem 9.1** For every mountain range that satisfies the two necessary conditions, it is possible for the mountain climbers to succeed.

We turn the problem into one about graphs. Imagine a vertical line drawn through the range at $M$. Put a dot at each peak and valley to the left of $M$ (including at $A$ and $M$). For each dot $d$, draw a horizontal line through $d$, and put new dots at each of the places where the horizontal line intersects the mountain range to the right of $M$ (there must be one such place, as on both sides the range ranges from sea-level to the level of $M$). For each such new dot $e$, create a vertex labelled $(d, e)$. The vertices represent each of the possible places where the left-hand climber has to change direction (from going up to going down, and vice-versa),
and all the possible places where the right-hand climber could be at that time. Do the same for each peak and valley to the right of $M$ (including at $Z$). The figure below shows all of the dots created in our running example. The nine vertices created are:

$$AZ, CT, CV, CX, DU, DY, MM, EW, BU.$$ 

Now put an edge between two vertices, say $ab$ and $cd$ if the following is true: if at some moment the climber on the left is at $a$ and the climber on the right is at $b$, there is a choice of direction (up or down) such that if the two climbers head in that direction, going at the right speeds to stay at the same altitude, the left climber will reach $c$ and the right climber will reach $d$, and in the process they won’t have passed through any other pair of points that forms a vertex.

For example, if the left climber is at $C$ and the right at $V$, the left climber could go down back towards $A$, and after a while would reach $B$ when the right climber reaches $U$; or he could go down forward towards $D$, and after a while would reach $D$ when the right climber reaches $U$. So vertex $CV$ is adjacent to both $BU$ and $DU$. The full graph is shown in the figure below.

Notice that one of the vertices of the graph is $AZ$, which represents the starting point for the two climbers, and another is $MM$, which represents the finishing point. If we could find a path in the graph from $AZ$ to $MM$, then that would gives the mountain climbers a recipe for succeeding.
The degree of $AZ$ in the graph is 1: both climbers have to go up, and each has only one up route to choose. Also, the degree of $MM$ is 1: both climbers have only the option of down, and they each have only one route they can take while staying on their side of the range.

All other vertices have degree either 2, 4 or 0: if the left climber is at a peak, and the right climber is on a slope (e.g., vertex $CT$), then the left climber has to choose one of two routes, and the right climber has no choice, leading to two neighbors. More generally the same is true if one climber is at either a peak or a valley and the other is on a slope. If both climbers are at a peak, or both at a valley, (e.g., vertex $DU$), then they each have two choices, leading to four neighbors. If one is at a peak and the other at a valley, then they are stuck, and the vertex has degree 0.

Consider the component of the graph that includes $AZ$. By the handshake lemma, that component must have at least one other odd-degree vertex. But $MM$ is the only possible other odd degree vertex. It follows that $AZ$ and $MM$ must be in the same component, so there is a path between them. In our running example, such a path is $AZ$ to $CX$ to $DY$ to $EW$ to $DU$ to $MM$.

Graph theory has saved the mountain climbers.

All pictures in this section have been taken from the beautiful article “The parallel climbers puzzle” by A. Tucker, that appeared in *Math Horizons* pages. 22–24 in 1995.
1. Is there a graph $G$ on nine vertices with three vertices have degree 1, two having degree 2, one each having degrees 3 and 4, and two having degree 5?

**Solution:** If there was such a graph, it would have score $(1, 1, 1, 2, 2, 3, 4, 5, 5)$. By the Hakami-Havel, such a graph exists if and only if there is a graph with score $(1, 1, 1, 1, 2, 3, 4)$ (delete largest number $a$, reduce $a$ largest remaining numbers each by 1), which exists if and only if there is a graph with score $(1, 1, 1, 0, 0, 1, 2)$, which exists if and only if there is a graph with score $(1, 1, 0, 0, 0, 0)$, which exists if and only if there is a graph with score $(0, 0, 0, 0)$. Such a graph does exist (the graph on five vertices with no edges), so a graph with three vertices have degree 1, two having degree 2, one each having degrees 3 and 4, and two having degree 5 also exists.

2. How many graphs are there on vertex set $\{1, 2, 3, 4, 5, 6, 7, 8\}$ that are isomorphic to the graph with edge set $\{\{1, 2\}, \{3, 4\}, \{5, 6\}, \{7, 8\}\}$? **Hint:** don’t try to draw them all; try to count them.

**Solution:** Such a graph is obtained exactly by splitting $\{1, 2, 3, 4, 5, 6, 7, 8\}$ into four block of size 2 each, without regard for order of blocks or order within blocks, and then joining each pair in a block by an edge.

There are $\binom{8}{2,2,2,2}$ (or, equivalently, $\binom{8}{2}\binom{6}{2}\binom{4}{2}\binom{2}{2}$) ways of splitting $\{1, 2, 3, 4, 5, 6, 7, 8\}$ into four block of size 2 each, without regard for order within blocks. But this way of counting puts an order on the blocks (there is a “first” block of size 2, a “second”, and so on). To remove this order on the blocks we divide by $4!$, the number of ways or ordering four objects.

The final count is $\binom{8}{2,2,2,2}/4! = 105$. 


11 Midsemester exam

Name:________________

1. (a) By using any algebraic expression that you know for the binomial coefficients, verify that
\[ n \binom{n-1}{k-1} = k \binom{n}{k}. \]

Solution:
\[ n \binom{n-1}{k-1} = n \frac{(n-1)!}{(k-1)((n-1)-(k-1))!} = \frac{n!}{(k-1)!(n-k)!} = \frac{k \cdot n!}{k!(n-k)!} = k \binom{n}{k}. \]

(b) Give a combinatorial proof of the identity. That is, describe clearly a set, which, if counted one way has size \( n \binom{n-1}{k-1} \) and, if counted another way has size \( k \binom{n}{k} \).

Solution: Consider the set of committees-with-a-chair of size \( k \) selected from a set of \( n \) people. One way to count the size of this set is to consider first selecting the chair (\( n \) options), then selecting the remaining \( k-1 \) members of the committee from among the remaining \( n-1 \) people (\( \binom{n-1}{k-1} \) options), leading to a total of \( n \binom{n-1}{k-1} \) selections. Another way to count the size of this set is to consider first selecting the committee (\( \binom{n}{k} \) options), then selecting the chair from among the \( k \) committee members (\( k \) options), leading to a total of \( k \binom{n}{k} \) selections.

2. (For this question, any (correct) expression involving factorials, binomial coefficients, multinomial coefficients, Stirling numbers, etc., is fine; no need to simplify). I have a class of \( 3n \) students, and I want to break them into three groups of size \( n \) each. In how many ways can I do this if

(a) I don’t care about the order in which I choose the groups, and I don’t care about order within the groups.

Solution: Many correct expressions, including
\[ \binom{3n}{n,n,n}/3!. \]
Note that \( \binom{3n}{n,n,n} \) counts the number of ways of breaking \( 3n \) people into a first group of size \( n \), a second of size \( n \) and a third of size \( n \), with order within the groups not mattering; dividing by \( 3! \) removes the order among the groups.

(b) I care about the order in which I choose the groups, but I don’t care about order within the groups.

Solution: Many correct expressions, including
\[ \binom{3n}{n,n,n}. \]

(c) I care both about the order in which I choose the groups, and order within the groups.

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Solution: Many correct expressions, including

\[
\binom{3n}{n,n,n}^3(n!)^3.
\]

The factor of \((n!)^3\) accounts for all the ways of putting order within each of the partitions counted in part (b).

3. (a) State the inclusion-exclusion formula for calculating \(|A \cup B \cup C \cup D|\).

Solution:

\[
|A \cup B \cup C \cup D| = |A| + |B| + |C| + |D| - |A \cap B| - |A \cap C| - |A \cap D| - |B \cap C| - |B \cap D| - |C \cap D|
\]

\[
= + |A \cap B \cap C| + |A \cap B \cap D| + |A \cap C \cap D| + |B \cap C \cap D|
\]

\[
= - |A \cap B \cap C \cap D|.
\]

(b) In how many of the 40 320 permutations of \(\{1, \ldots, 8\}\) is no even number is mapped to itself?

Solution: Let \(A_2\) be the set of permutations with 2 mapped to itself, \(A_4\) be the set of permutations with 4 mapped to itself, \(A_6\) be the set of permutations with 6 mapped to itself, and \(A_8\) be the set of permutations with 8 mapped to itself.

Note that \(|A_2| = |A_4| = \ldots = 7!\), \(|A_2 \cap A_4| = |A_2 \cap A_6| = \ldots = 6!\), \(|A_2 \cap A_4 \cap A_6| = |A_2 \cap A_4 \cap A_8| = \ldots = 5!\), and \(|A_2 \cap A_4 \cap A_6 \cap A_8| = 4!\) (think about how many elements are free to go any where in each case), so by inclusion-exclusion

\[
|A_2 \cup A_4 \cup A_6 \cup A_8| = 4 \times 7! - 6 \times 6! + 4 \times 5! - 4!,
\]

and the number of permutations with no even number mapped to itself is

\[
40320 - (4 \times 7! - 6 \times 6! + 4 \times 5! - 4!) = 24024.
\]

4. (a) For the graph \(G\) drawn below, write down an \(a-d\) walk that is not a path.

Solution: Any path that repeats a vertex, such as \((a, e, d, b, d)\) (or \((a, e, a, e, d, d, b, b, d, d)\)) works.

(b) The trace of an \(n\) by \(n\) matrix \(A\), denoted by Tr\((A)\), is the sum of the entries down the main diagonal (so for example \(\text{Tr} \begin{pmatrix} 2 & 1 \\ 5 & 7 \end{pmatrix} = 9\)). Show that for any graph with \(m\) edges and with adjacency matrix \(A\), \(\text{Tr}(A^2) = 2m\).
Solution: The $ii$ entry of $A^2$ counts the number of length 2 walks from $i$ to $i$, which is exactly the number of edges leaving $i$ (such a walk goes out along one such edge, and back along the same edge), which is the degree of $i$. So the trace of $A^2$ is the sum of the degrees of the graph, which is twice the number of edges, or $2m$.

5. (a) The ACC has 11 member universities. Is it possible for the 11 ACC football teams to organize an annual conference schedule in which each of the 11 plays exactly 5 other ACC teams?

Solution: Suppose it were possible. Encode the schedule as a graph, with the universities as vertices and two vertices adjacent if those two play each other. Such a graph has all degrees 5, so sum of degrees 55, impossible since the sum of degrees must be even. So it is not possible.

(b) Let $G$ be a graph with $n$ vertices and $n-1$ edges. Show that $G$ must have at least one vertex with degree less than 2.

Solution: Suppose all degrees are at least 2. Then the sum of the degrees is at least $2n$. But the sum of the degrees is twice the number of edges, or $2n-2$. This is a contradiction. So some degrees must be at most 1.

6. (a) In how many ways can the Cubs and the Mets play a best-of-seven game series (i.e., first team to win four games wins series)? [I only care about the order in which teams win games, so for example if it was a best-of-three series, there would six be ways: three in which the Cubs win (CC, CMC, MCC) and three in which the Mets win (MM, MCM, CMM)]

Solution: Suppose Mets win series. The they must win the last game, and any three of the earlier games. If the series goes $k$ games, then, there are $\binom{k-1}{3}$ ways in which the Mets can win. Since $k$ ranges from 4 to 7, and we must also consider the possibility of the Cubs winning, the final count is

$$2 \left( \binom{3}{3} + \binom{4}{3} + \binom{5}{3} + \binom{6}{3} \right) = 70.$$  

(b) How many solutions are there to the equation $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 = 0$, if each $x_i$ must be one of $-1, 0$ or $+1$?

Solution: There must be an equal number of $+1$’s as $-1$’s, and this number is one of 0, 1, 2 or 3. With $k$ $1$’s and $k$ $-1$’s, there are $\binom{k}{2k}$ ways to choose the locations of the $+1$’s and $-1$’s, and $\binom{2k}{k}$ ways to choose which of these are $+1$’s. The final count is

$$\binom{7}{0} \binom{0}{0} + \binom{7}{2} \binom{2}{1} + \binom{7}{4} \binom{4}{2} + \binom{7}{6} \binom{6}{3} = 393.$$
12 Quiz 2

This quiz is about closed Eulerian trails. In the textbook (Section 4.4 and later) these are referred to a closed Eulerian tours.

1. The town of Königsberg, with its seven bridges and four land masses (A, B, C and D), is shown below.

The town wants to build some more bridges, in order to make it possible for someone to travel around the town, crossing each bridge once and only once, ending up back where they started. If they want to build the minimum possible number of extra bridges, which pairs of land masses should be joined?

**Solution:** The multi-graph on vertex set A, B, C, D in which there are as many edges between two vertices as there are bridges between the corresponding land-masses has all vertices having degree 3. In order for the bridge problem to be solvable, it is necessary to add bridges to make all degrees be even (the multigraph is already connected, so all we need to ensure a closed Eulerian trail is that all degrees be even). Clearly two edges/bridges at least are needed to get all vertices up to even degree, as each edge/bridge can change the degree of at most two vertices. Any pair of two edges/bridges that between them use all four land-masses will work; A to B and C to D will work; as will A to C and B to D, and A to D and B to C.

2. A certain graph $G$ has a closed Eulerian trail. I find a cycle in $G$, and remove the edges of that cycle from $G$ to form a new graph $G'$. EITHER prove that $G'$ has a closed Eulerian trail, OR give an example to show that it sometimes doesn’t have one.

**Solution:** The result sometimes may not have a closed Eulerian trail. For example, if $G$ has vertex set \{1, 2, 3, 4, 5, 6, 7, 8, 9, 0\}, with edges 12, 23, 34, 41, 45, 56, 67, 74, 68, 89, 90, 06, the $G$ has a closed Eulerian trail. But if I remove the edges 45, 56, 67, 74 (the edges of a cycle), the result graph is disconnected and so does not have a closed Eulerian trail (though each component of the resulting graph has a closed Eulerian trail).
13 Homework 6— due Thursday November 5

Reading for Homework 6

• Sections 4.4, 4.5, 5.1.

Problems

3 pts Construct two connected graphs with the same score, one with and one without a Hamiltonian cycle. (See Section 4.4, exercise 7 (page 136), for a definition of Hamiltonian cycle.)

Solution: Lots of possible examples; maybe the simplest is on six vertices \{a, b, c, d, e, f\}. On the one hand, join these six in a cycle (so there is a Hamiltonian cycle), and add one more edge joining two vertices on opposite sides of the cycle, so the score is \(2, 2, 2, 2, 3, 3\). On the other hand join \(a, b, c\) in a triangle, and \(d, e, f\) in a triangle, and join \(a\) to \(d\). We still have score \(2, 2, 2, 2, 3, 3\), but now it’s easy to see that we don’t have a Hamiltonian cycle.

4 pts For a graph \(G = (V, E)\), the line graph of \(G\), denoted \(L(G)\), is the graph whose vertex set is \(E\), the set of edges of \(G\), and with \(e\) and \(e'\), two edges of \(G/\)vertices of \(L(G)\), adjacent in \(L(G)\) exactly when \(e\) and \(e'\) share a common vertex in \(G\). For example, if \(G\) is the path on vertex set \(\{v_1, v_2, v_3, v_4\}\) with edges \(e_1 = v_1v_2, e_2 = v_2v_3\) and \(e_3 = v_3v_4\), then \(L(G)\) has vertices \(e_1, e_2, e_3\), with \(e_1\) joined to \(e_2\) and \(e_2\) joined to \(e_3\).

Decide whether each of the following two statements are true or false, with justifications:

(a) \(G\) is connected if and only if \(L(G)\) is connected.

Solution: It is true that if \(G\) is connected then \(L(G)\) is connected, but the converse is false. Consider, for example, \(G\) a triangle together with an isolated vertex. Then \(L(G)\) is a triangle, so connected, but \(G\) is not connected.

(b) \(G\) has a closed Eulerian trail if and only if \(L(G)\) has a Hamiltonian cycle.

Solution: It is true that if \(G\) has a closed Eulerian trail then \(L(G)\) has a Hamiltonian cycle, but the converse is false. Consider, for example, \(G\) a star with three edges and four vertices (one vertex of degree 3, the rest of degree 1). Then \(L(G)\) is a triangle, so has a Hamiltonian cycle, but \(G\) does not have a closed Eulerian trail.

4 pts This question is about graphs being randomly Eulerian (or not). See Section 4.4, exercise 10 (page 137) for the definition.

(a) For the graph below, explain clearly why it is randomly Eulerian.
Solution: Consider a maximal trail starting at $v_0$. It must end at $v_0$, since when we enter any other vertex, the number of available edges is the vertex’s degree, minus one (for the edge we have just come in on), minus an even number (twice the number of times we have previously visited that vertex), so it is odd, so it can’t be zero. Because the trail is maximal, we must have used all edges out of $v_0$ when we stop. Remove the edges of the maximal trail. In any non-trivial component of the resulting graph (one with at least one edge) all vertex degrees are even (we have taken even degrees and removed an even number of edges from each one), and so, as we have seen previously, each such component has a cycle. But $G$ has no cycles that do not include $v_0$, and the component of $v_0$ is trivial, so we conclude that there are no non-trivial components. So we are left with a graph which has no edges, and our maximal trail was an Eulerian trail, proving that the graph is randomly Eulerian.

(b) For the graph below, explain clearly why it is not randomly Eulerian.
Solution: $G$ has cycles which do not pass through $v_0$. Remove one of them. The result is a connected graph (easy to see) with all degrees still even (again easy), so the graph has a closed Eulerian trail. Such a trail is a maximal trail in the original graph that does not cover all the edges, so the original graph is not randomly Eulerian.

3 pts Let $G$ be a graph on $n$ vertices, $n \geq 1$. Prove that the following are equivalent:

(a) $G$ is a tree;
(b) $G$ has no cycles, and has $n - 1$ edges.

[You can assume every statement already proven in Section 5.1].

Solution: If $G$ is a tree then by definition it has no cycles, and by a theorem from class it has $n - 1$ edges, so a) implies b).

If $G$ has no cycles then it is a forest. Suppose it has $k$ components, $C_1, ..., C_k$. We can make it into a tree by adding any edge from $C_1$ to each of $C_2$ through $C_k$, so $k - 1$ new edges. This tree has $n - 1 + k - 1$ edges, but also, by a theorem from class it has $n - 1$ edges. So $k = 1$, $G$ is connected and so is a tree, and b) implies a).

3 pts Give a proof that if a tree $T$ has a vertex of degree $k$, then it must have at least $k$ leaves.

Solution: Let $v$ be a vertex of degree $k$, with neighbors $w_1, ..., w_k$. Consider a longest path in $T$ that has $v$ as an endvertex, and starts with the edge $vw_1$. As we argued in class, the other endvertex of this path, $x_1$ say, must be a leaf (note that $x_1$ may equal $v_1$). Similarly we have leaves $x_2, ..., x_k$. No two of $x_1, ..., x_k$ can coincide, since if they did we would have a closed walk and hence a cycle in $T$. So $T$ has at least $k$ leaves.
14 Quiz 3 Name: ______________________

1. Write down the Prüfer code of this tree:

\[
\begin{array}{c}
1 \\
2 \quad 3 \\
4 \quad 5 \quad 6 \\
8 \quad 9 \\
10 \\
11 \\
12 \\
13 \\
14 \\
15 \\
\end{array}
\]

Solution: \((4, 4, 2, 5, 5, 2, 1, 3, 6, 6, 3, 7, 7)\).

2. A tree on vertex set \(\{1, \ldots, n\}\) (for some \(n\)) has Prüfer code \((2, 2, 4, 4, 6)\). Say what \(n\) is, and then draw the tree (with vertices clearly labeled).

Solution: 2 joined to 1, 3, 4; 4 also joined to 5, 6; 6 also joined to 7.

3. How many non-identical trees are there on vertex set \(\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}\) in which vertex 10 has degree exactly 5?

Solution: We count the number of different Prüfer codes that such a tree can have. Such a code is a word of length 8 in alphabet of size 10 in which one letter, 10, appears 4 times. There are

\[
\binom{8}{4} 9^4
\]

such codes, and hence this many such trees. The \(\binom{8}{4}\) locates the 4 slots in the code where the letter 10 appears, and the \(9^4\) fills in the rest of the code.
15 Homework 7 — due Thursday November 19

Reading for Homework 7

- Sections 5.3, 5.4, 8.1, 8.2, 8.4.

Problems

1. Run Kruskal’s algorithm on the following (weighted) graph to find a minimum weight spanning tree. Clearly mark the edges of the tree you find on the picture, and calculate the weight of the tree.

![Graph Image]

**Solution:** The minimum weight spanning tree given by Kruskal’s algorithm uses the edges of weights 2, 3, 4, 5, 6, 7, 9, 10, and 11, for total weight 57.

2. Section 5.4, Exercise 2 (page 175). [Your answer should consist of two parts. First, show that $T + e$ contains at least one cycle. Then show that it cannot contain two or more cycles. You *should* use results from Section 5.1 for this problem. If you don’t, you will be re-inventing the wheel. If in the course of your solution you re-prove a result from Section 5.1, I’ll be mean and deduct points.]

**Solution:** A tree $T$ is *maximally acyclic*, so adding any edge to $T$ creates at least one path. If two cycles, $C_1, C_2$ say, were created by adding the edge $xy$ to $T$, there would be two paths in $T$ from $x$ to $y$, one following $C_1$ and the other following $C_2$, violating the fact that in a tree there is a *unique path* between any two vertices; so the addition of a single edge adds exactly one cycle.

3. Section 5.4, Exercise 4 (page 175). [This addresses a question that was asked at the end of class on Thursday, about whether the minimum spanning tree is unique. It’s not always, but here is one circumstance under which it is, namely, when all weights are different. Again, you *should* use results from Section 5.4, to streamline your proof.]
Solution: Let $T = \{e_1, \ldots, e_m\}$ be a tree produced by Kruskal (which is unique since with distinct edge weights, Kruskal never has to make an arbitrary choice), with $w(e_1) < w(e_2) < \ldots < w(e_m)$ (note $<$ rather than $\leq$ since weights are distinct). Let $T' = \{e'_1, \ldots, e'_m\}$ be any other spanning tree, also with $w(e'_1) < w(e'_2) < \ldots < w(e'_m)$. In class we proved that $w(e_i) \leq w(e'_i)$ for each $i$. If $T'$ is different from $T$, then there must be at least one $i$ with $w(e_i) \leq w(e'_i)$ (if $w(e_i) = w(e'_i)$ for all $i$, then $e_i = e'_i$ for all $i$ (by injectivity of $w$), so $T = T'$). So $w(e_i) \leq w(e'_i)$ for each $i$, and at least once the inequality is strict. It follows that $\sum_{i=1}^m w(e_i) \leq \sum_{i=1}^m w(e'_i)$, or $w(T) < w(T')$. So every spanning tree different from the unique one produced by Kruskal has greater weight than Kruskal’s tree, and there is a unique minimum spanning tree.

4. Section 5.4, Exercise 11, part a) (page 176). [This exercise explores a situation where a greedy algorithm is not necessarily always optimal, but is always close to optimal.]

Solution: The smallest edge cover has size at least $n/2$, since each added edge can cover at most two new vertices. The greedy algorithm has to terminate after at most $n - 1$ steps, since at the first step two new vertices get covered, and at all subsequent steps at least one new vertex gets covered. Since $n - 1 \leq 2(n/2)$, the greedy algorithm will always produce an edge cover that is no more than twice the size of the smallest possible edge cover.

5. For $n \geq 4$ and $3 \leq k \leq n - 1$. An $(n, k)$-star-path is a tree with the following structure: it has a single vertex of degree $k$, and radiating out from that vertex are $k$ paths ending in $k$ leaves (the picture below shows a $(6, 3)$-star-path — ask me if the definition is unclear!).

How many non-identical $(n, k)$-star-paths are there on vertex set $\{1, \ldots, n\}$?

Solution: We count the number of different Prüfer codes that an $(n, k)$-star-path can have. Such a code is a word of length $n - 2$ in alphabet of size $n$ in which one letter appears $k - 1$ times (the center of the star-path), $k$ letters appear 0 times (the $k$ leaves), and all other letters appear once (the $n - k - 1$ vertices of degree 2). There are

$$n \binom{n - 1}{k} \binom{n - 2}{k - 1} (n - k - 1)!$$

such codes, and hence this many $(n, k)$-star-paths. The $n$ is for the choice of letter to appear $k - 1$ times, the $\binom{n - 1}{k}$ is for the choice of $k$ letters to appear 0 times, the $\binom{n - 2}{k - 1}$ locates the $k - 1$ slots in the code where the letter that appears $k - 1$ times actually appears, and the $(n - k - 1)!$ fills in the rest of the code with one occurrence of each of the remaining $n - k - 1$ letters.

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