Basic Combinatorics
Math 40210, Section 01 — Spring 2012
Homework 7 — due Monday, March 26

General information: I encourage you to talk with your colleagues about homework problems, but your final write-up must be your own work.
You should present your final homework solutions clearly and neatly. Keep in mind that when you write a homework solution, you are trying to communicate the solution to someone other than yourself, so incomplete sentences and personal shorthand is not helpful!
Due to manpower issues, I will only grade selected homework problems, but I plan to quickly post solutions to all the problems soon after I’ve collected them up.

Reading:

- Introduction to Chapter 2
- Section 2.1
- Section 2.2
- Section 2.3

Problems:

- Section 2.1: 1, 3, 6, 11
- Section 2.2: 2, 3, 4, 7(d,e), 8 (just for falling powers)
- Section 2.3: 2, 7, 9(d,e), 10. (For problem 2, it should be clarified that the steps must always be taken in a positive direction: you can go from \((x, y, z)\) to any of \((x + 1, y, z)\), \((x, y + 1, z)\) or \((x, y, z + 1)\), but not for example to \((x − 1, y, z)\).)
Solutions:

- **2.1 1a**: 53 choice for initial character, 63 for all the rest, so \((53)(63)(63)(63)(63)\) in total.

- **2.1 1b**: 53 with one character, \(53 \times 63\) with two characters, so \((53)+(53)(63)+(53)(63)(63)+(53)(63)(63)(63)(63)\) in total.

- **2.1 1c**: 53 with one character, 53 with two characters (first character determines second), \(53 \times 63\) with three characters (first character determines third), \(53 \times 63\) with four characters (first two characters determine third and fourth), \(53 \times 63\times 63\) with five characters (first two characters determine fourth and fifth), so \((53) + (53) + (53)(63) + (53)(63) + (53)(63)(63)\) in total.

- **2.1 3a**: \(30!\)

- **2.1 3b**: \((14)(13)(12)\)

- **2.1 3c**: \((\binom{15}{8}) \times (\binom{15}{8})\)

- **2.1 3d**: In the western division there are in total 45 centers from which three must be chosen, 60 guards from which four must be chosen, and 75 forwards from which five must be chosen, leading to a total of \((\binom{45}{3}) (\binom{60}{4}) (\binom{75}{5})\).

- **2.1 6**: I think that this question has some ambiguities!

  Each multiple choice question has \(2^4 = 16\) possible answers (you can choose an arbitrary subset of \(\{a, b, c, d\}\)). There are 10 such questions, so \(16^{10}\) possibilities for the multiple choice part.

  Each T/F question has 2 possible answers (one is required to give an answer), so \(2^8\) possibilities for the true/false part.

  In the definition section, EITHER for each of seven terms, there is a set of 10 definitions from which one must be chosen, leading to a total \(10^7\) possibilities for this part, OR there is a set of seven terms and 10 definitions, and you have to match the terms into the definitions, in which case there are \(10^7\) for this part (assuming that no definition works for multiple terms).

  I don’t understand the last instruction. EITHER it means that you do the multiple choice, and then do either the T/F or the definitions, in which case there are \(16^{10}(2^8 + 10^7)\) (or \(16^{10}(2^8 + 10^2)\)) possibilities in total, OR it means that you do one of the multiple choice, the T/F or the definitions, in which case there are \(16^{10} + 2^8 + 10^7\) (or \(16^{10} + 2^8 + 10^2\)) possibilities in total.

- **2.1 11**: An \(n\) which is a positive integer divisor of \(N\) has prime factorization \(p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_m^{\alpha_m}\), where each \(\alpha_i\) satisfies \(0 \leq \alpha_i \leq n_i\). So there are \(n_i + 1\) choices for each \(\alpha_i\), leading to \((n_1 + 1)(n_2 + 1)\ldots(n_m + 1)\) distinct positive integer divisors of \(N\).

- **2.2 2**: An algebraic proof is easy:

\[
\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n}{k} \frac{(n-1)!}{(k-1)!(n-k)!} = \frac{n}{k} \frac{(n-1)!}{(k-1)!\left((n-1)-(k-1)\right)!} = \frac{n}{k} \binom{n-1}{k-1}. \]
For a combinatorial proof, notice that multiplying through by \( k \) one gets

\[
k \binom{n}{k} = n \binom{n-1}{k-1}.
\]

This is exactly the committee-chair identity from Quiz 4 (the online solution for which gives a counting proof).

**2.2 3:** We have

\[
\binom{n}{k} \binom{k}{m} = \frac{n!}{k!(n-k)!} \frac{k!}{m!(k-m)!} = \frac{n!}{(n-k)!m!(k-m)!} = \frac{1}{m!(k-m)!(n-k)!}
\]

2.2 4:** Either he first selects the \( k \) paintings to display from the \( n \), and then selects the \( m \) paintings from the \( k \) to display prominently, leading to a count of

\[
\binom{n}{k} \binom{k}{m},
\]

or he first chooses the \( m \) paintings from the \( n \) to display prominently, then chooses the remaining \( k-m \) from the remaining \( n-m \) to also display (but less prominently), leading to a count of

\[
\binom{n}{m} \binom{n-m}{k-m}.
\]

Notice that this is a generalization of the committee-chair identity from Quiz 4 (which is the case \( m = 1 \)).

**2.2 7d:** This is a tricky one! Without using \( \binom{n}{k} = \binom{n}{n-k} \), I know of no way to approach this (no algebraic or inductive proof, for example). Using \( \binom{n}{k} = \binom{n}{n-k} \), we have

\[
\sum_k \binom{n}{k}^2 = \sum_k \binom{n}{k} \binom{n}{n-k}.
\]

The right hand-side is counting the number of ways of selecting a set of size \( n \) from a set \( \{a_1, \ldots, a_n, b_1, \ldots, b_n\} \) of size \( 2n \), by first deciding how many of the \( n \) comes from the \( a_i \)'s (\( k \) of them, leading to a count of \( \binom{n}{k} \)), forcing the remainder to comes from the \( b_i \)'s (\( n - k \) of them, leading to a count of \( \binom{n}{n-k} \)). But by a direct count, we get that this is just \( \binom{2n}{n} \). (Notice
that this is an example of the vandermonde convolution from page 142, with $m = \ell = n$ in the displayed equation above (2.11). In summary:

$$\sum_k \binom{n}{k}^2 = \binom{2n}{n}.$$ 

- **2.2 7e**: If $n = 0$ and $m$ is negative, then the sum is $0$ (it is empty). If $n = 0$ and $m \geq 0$, then the sum is $1$. That deals with $n = 0$; so from now on we assume $n \geq 1$.

For $n \geq 1$, if $m < 0$, then the sum is $0$ (it is empty). For $m = 0$, there’s just one term, and the sum is $1$. For $m \geq n$, the sum is the same as if we stopped at $n$, so it’s $0$, as we proved in class. So the remaining (and most interesting) cases are $n \geq 2$ and $1 \leq m \leq n - 1$.

A little experimentation with Pascal’s triangle suggests that in this range:

$$\sum_{k \leq m} (-1)^k \binom{n}{k} = (-1)^m \binom{n-1}{m}.$$ 

For each fixed $n \geq 2$, we prove this by induction on $m$, with the case $m = 0$ trivial. For $m > 0$ we have

$$\sum_{k \leq m} (-1)^k \binom{n}{k} = \binom{n}{m} + \sum_{k \leq m-1} (-1)^k \binom{n}{k}$$

$$= (-1)^m \binom{n}{m} + (-1)^{m-1} \binom{n-1}{m-1}$$

$$= (-1)^m \left( \binom{n}{m} - \binom{n-1}{m-1} \right)$$

$$= (-1)^m \binom{n-1}{m}$$

the second equality using the induction hypothesis and the last equality using Pascal’s identity.

- **2.2 8**: I don’t know of an easy way to do this algebraically or by induction. If we allow ourselves to only prove the result for $x, y$ positive integers, then there are two easy approaches:

First, we can write

$$\binom{n}{k} x^k y^{n-k} = \frac{n!}{k!(n-k)!} x^k y^{n-k}$$

$$= \frac{n!}{k! (n-k)!} x^k y^{n-k}$$

$$= n! \binom{x}{k} \left( \frac{y}{n-k} \right).$$

So, by vandermonde’s convolution, we have

$$\sum_k \binom{n}{k} x^k y^{n-k} = \sum_k n! \binom{x}{k} \left( \frac{y}{n-k} \right) = n! \sum_k \binom{x}{k} \left( \frac{y}{n-k} \right) = n! \binom{x+y}{n}.$$
But also

\[(x + y)^n = n! \binom{x + y}{n}.\]

So we have the identity.

Here’s another, more combinatorial way: the left hand side directly counts the number of ways of taking \(n\) elements from a set of size \(x + y\), say \(\{a_1, \ldots, a_x, b_1, \ldots, b_y\}\), and arranging the \(n\) elements in order. Another way to do this is to select \(k\) elements (\(k\) running from 0 to \(n\)) from \(\{a_1, \ldots, a_x\}\) and arrange them in order (\(x^k\) ways to do this), take \(n - k\) elements from \(\{b_1, \ldots, b_y\}\) and arrange them in order (\(y^{n-k}\) ways to do this), and then merge the two ordered sets to get an ordered list of \(n\) elements from the full set of \(a\)’s and \(b\)’s (\(\binom{n}{k}\) ways to do this - just choosing the \(k\) slots into which the \(a\)’s go). So the right hand side also counts the number of ways of taking \(n\) elements from a set of size \(x + y\) and arranging them in order.

• 2.3 2: In order to reach \((a, b, c)\) from \((0, 0, 0)\) taking steps parallel to (and in the same direct as) \((1, 0, 0)\), \((0, 1, 0)\) and \((0, 0, 1)\), we need to take exactly \(a + b + c\) steps. \(a\) of these steps must be steps of the form \((1, 0, 0)\), \(b\) of them must be of the form \((0, 1, 0)\), and \(c\) of them must be of the form \((0, 0, 1)\). So we completely determine a path by partitioning the set \(\{1, \ldots, a+b+c\}\) into three classes, class 1 of size \(a\), class 2 of size \(b\) and class 3 of size \(c\), with \(i\) falling into class 1 indicating that the \(i\)th step is of the form \((1, 0, 0)\), etc.. There are exactly \(\binom{a+b+c}{a,b,c}\) such partitions.

• 2.3 7: The left hand side counts the number of ways of partitioning a set of size \(m + n\), say \(\{a_1, \ldots, a_m\} \cup \{b_1, \ldots, b_n\}\) into three classes, the first of size \(a\), the second of size \(b\) and the third of size \(c\). The count is direct.

Another (indirect) way to count the same thing is to first decide how many of the \(a_i\)’s go into class 1 (say \(\alpha\) of them), how many of the \(a_i\)’s go into class 2 (say \(\beta\) of them), and how many of the \(a_i\)’s go into class 3 (say \(\gamma\) of them), then count the number of partitions that actually achieve this split (the summand of the right hand side counts exactly this: if \(\alpha\) of the \(a_i\)’s go into class 1, then \(\alpha - \alpha\) of the \(b_i\)’s must, etc.), then sum this quantity over all possible choices of \(\alpha, \beta\) and \(\gamma\) (for which the only constraint is \(\alpha + \beta + \gamma = m\), since all of the \(a_i\)’s must go into some class). This is exactly the right-hand side.

NB: I’m not vouching for the 100% accuracy of the numbers from here on - please let me know if you spot errors!

• 2.3 9d: We have 6 A’s, 2 K’s, 2 L’s, 2 S’s, 1 N, and 1 U.
  
  – \(r = 3\): Total 181.
    * 1 choice for word type \(xxx\), each with 1 anagram;
    * 20 choices for word type \(xxy\), each with 3 anagrams;
    * 20 choices for word type \(xyz\), each with 6 anagrams.
  
  – \(r = 4\): Total 897.
    * 1 choice for word type \(xxxx\), each with 1 anagram;
    * 5 choices for word type \(xxxy\), each with 4 anagrams;
* 6 choices for word type \(xyyy\), each with 6 anagrams;
* 40 choices for word type \(xyyz\), each with 12 anagrams;
* 15 choices for word type \(xyzw\), each with 24 anagrams.

- \(r = 14\):

\[
\frac{14!}{6!2!2!2!} = 14!
\]

- \(r = 4\): Total 80
  * 2 choices for word type \(xxxx\), each with 1 anagram;
  * 6 choices for word type \(xxxy\), each with 4 anagrams;
  * 3 choices for word type \(xxyy\), each with 6 anagrams;
  * 3 choices for word type \(xxyz\), each with 12 anagrams;
  * 0 choices for word type \(xyzw\), each with 24 anagrams.

- \(r = 5\): Total 201
  * 1 choice for word type \(xxxxx\), each with 1 anagram;
  * 4 choices for word type \(xxxxy\), each with 5 anagrams;
  * 3 choices for word type \(xxxxy\), each with 10 anagrams;
  * 3 choices for word type \(xxxxz\), each with 20 anagrams;
  * 3 choices for word type \(xxxyz\), each with 30 anagrams;
  * 0 choices for all other word types.

- \(r = 12\):

\[
\frac{12!}{5!4!3!} = 12!
\]

- \(r = 12\): The extra words that can be formed from the letters of Bobo, Mississippi are exactly the words with either one or two B’s. First, the count for words with 2 B’s:

  - 0 choices for word type \(xxxx\);
  - 0 choices for word type \(xxxy\);
  - 4 choices for word type \(xxyy\), each with 6 anagrams;
  - 10 choices for word type \(xxyz\), each with 12 anagrams;
  - 0 choices for word type \(xyzw\).

This gives a count of 144. Next, the count for words with 1 B:

  - 0 choices for word type \(xxxx\);
  - 2 choices for word type \(xxxy\), each with 4 anagrams;
  - 0 choices for word type \(xxyy\), each with 6 anagrams;
  - 16 choices for word type \(xxyz\), each with 12 anagrams;
  - 10 choices for word type \(xyzw\), each with 24 anagrams.

This gives a count of 440. The grand total, 584, is greater than 267