BASIC COMBINATORICS (MATH 40210) SEC 01, SPRING 2012, QUIZ 4

SOLUTIONS

(1) I take out two Aces from a standard deck of 52 cards. How many ways are there to select three more cards from the remaining 50, in such a way that the five cards together form a full house (3 of one kind, 2 of another)? (**Hint**: you may want to break your analysis into cases, depending on how many Aces you end up with in total)

Solution: Either my full house consists of 3 Aces and 2 of another card, or 2 Aces and 3 of another card.

In the first case, there are $\binom{2}{1}$ ways to choose 1 Ace from the two that remain in the deck, then 12 ways to choose the value of the card in the pair, and then finally $\binom{4}{2}$ ways to choose 2 cards of that value from the possible 4, leading to a total of $\binom{2}{1}12\binom{4}{2}$ selections.

In the second case, there are 12 ways to choose the value of the card in the three-of-a-kind, and then $\binom{4}{3}$ ways to choose 3 cards of that value from the possible 4, leading to a total of $12\binom{4}{3}$ selections.

In total, the number of selections is

$$\binom{2}{1}12\binom{4}{2} + 12\binom{4}{3} = 192.$$

(2) A basic identity involving binomial coefficients is that $k\binom{n}{k} = n\binom{n-1}{k-1}$. Give a *combinatorial* proof of this identity, by showing that both sides of the identity count the same thing (in different ways). (**Hint**: count the number of ways of selecting a committee of k people from a group of n people, with one person on the committee being designated the chair)

Solution: One way to select a committee of k people from a group of n people, with one person on the committee being designated the chair, is to first choose the committee ($\binom{n}{k}$ choices) and then choose the chair from among the committee members (k choices), leading to a total of $k\binom{n}{k}$ choices.

Another way is to first choose the chair (n choices) and then choose the remaining $k-1$ members of the committee from the remaining $n-1$ people ($\binom{n-1}{k-1}$ choices), leading to a total of $n\binom{n-1}{k-1}$ choices.

Since both methods count the same thing, we conclude $k\binom{n}{k} = n\binom{n-1}{k-1}$.

Date: Wednesday, March 21.