Basic Combinatorics

Math 40210, Section 01 — Spring 2012

Notes on monotone sequences

A sequence of real numbers $a_1, a_2, a_3, \ldots$ is **monotone increasing** if $a_1 \leq a_2 \leq a_3 \ldots$, and **monotone decreasing** if $a_1 \geq a_2 \geq a_3 \ldots$. A sequence is **monotone** if it is either monotone increasing or monotone decreasing.

**Infinite monotone subsequences**

The following is an important and useful fact in analysis:

Every infinite real sequence contains an infinite subsequence which is monotone.

Here’s a proof of this fact, the repeatedly uses the following variant of the pigeon-hole principle: if infinitely many pigeons are distributed between two pigeon-holes, one of the pigeon-holes must contain infinitely many pigeons.

We begin by constructing a subsequence $b_1, b_2, b_3, \ldots$ using the following iterative procedure:

- Set $b_1 = a_1$.

- If there are infinitely many terms of the sequence $a_2, a_3, a_4, \ldots$ that are at least as large as $b_1$, then keep these terms, throw out all the terms that are less than $b_1$, and label $b_1$ UP (indicating that when you move from $b_1$ to a later term in the thinned sequence, you move up). If there are *not* infinitely many terms of the sequence $a_2, a_3, a_4, \ldots$ that are at least as large as $b_1$, then by the pigeon-hole principle there are infinitely many terms of the sequence $a_2, a_3, a_4, \ldots$ that are smaller than $b_1$. If this is the case, keep these terms, throw out all the terms that are at least as large as $b_1$, and label $b_1$ DOWN (indicating that when you move from $b_1$ to a later term in the thinned sequence, you move down).

- Whichever operation has happened, we now have an infinite subsequence $b_1, a_{i_1}, a_{i_2}, a_{i_3}, \ldots$, either with the property that $b_1 \leq a_{i_k}$ for each $k$ (if $b_1$ is labeled UP) or $b_1 > a_{i_k}$ for each $k$ (if $b_1$ is labeled DOWN).

- Set $b_2 = a_{i_1}$.

- If there are infinitely many terms of the sequence $a_{i_2}, a_{i_3}, a_{i_4}, \ldots$ that are at least as large as $b_2$, then keep these terms, throw out all the terms that are less than $b_2$, and
Here’s a finite version of the same fact, using essentially the same proof.

Finite monotone subsequences

Here’s a finite version of the same fact, using essentially the same proof.

Every sequence of length $2^{2n-2}$ contains a subsequence of length $n$ which is monotone.

The proof repeatedly uses the following variant of the pigeon-hole principle: if $m$ is an odd number, and $m$ pigeons are distributed between two pigeon-holes, one of the pigeon-holes must contain at least $(m + 1)/2$ pigeons (if not, each hole contains at most $(m + 1)/2 - 1$ pigeons, accounting for at most $m - 1$ pigeons in total).

We begin by constructing a subsequence $b_1, b_2, b_3, \ldots, b_{2n-1}$ using the following iterative procedure:

- Set $b_1 = a_1$.
- If there are $2^{2n-3}$ terms of the sequence $a_2, a_3, a_4, \ldots, a_{2^{2n-2}}$ that are at least as large as $b_1$, then keep these terms, throw out all the terms that are less than $b_1$, and label $b_1$ UP. If there are not $2^{2n-3}$ terms of the sequence $a_2, a_3, a_4, \ldots, a_{2^{2n-2}}$ that are at least as large as $b_1$, then by the pigeon-hole principle there are $2^{2n-2}$ terms of the sequence $a_2, a_3, a_4, \ldots, a_{2^{2n-2}}$ that are smaller than $b_1$. If this is the case, keep these terms, throw out all the terms that are at least as large as $b_1$, and label $b_1$ DOWN.

• Whichever operation has happened, we now have an infinite subsequence $b_1, b_2, a_{i_1}', a_{i_2}', a_{i_3}', \ldots$, with the property that either $b_1 \leq b_2$ and $b_1 \leq a_{i_k}'$ for each $k$ (if $b_1$ is labeled UP) or $b_1 > b_2$ and $b_1 > a_{i_k}'$ for each $k$ (if $b_1$ is labeled DOWN); AND $b_2 \leq a_{i_k}'$ for each $k$ (if $b_2$ is labeled UP) or $b_2' > a_{i_k}$ for each $k$ (if $b_2$ is labeled DOWN).

• Repeating this process infinitely many times, we eventually get an infinite subsequence $b_1, b_2, b_3, \ldots$, with each $b_i$ labeled either UP or DOWN, and with the property that if $b_i$ is labeled UP then $b_i \leq b_j$ for all $j > i$, and if $b_i$ is labeled DOWN then $b_i > b_j$ for all $j > i$.

The sequence $b_1, b_2, b_3, \ldots$ is not necessarily monotone. However, by one more application of our pigeon-hole principle, at least one of the labels UP, DOWN must occur infinitely often. If the label UP occurs infinitely often, then the infinite subsequence consisting of those $b_i$ labeled UP forms an infinite monotone increasing sequence; if the label DOWN occurs infinitely often, then the infinite subsequence consisting of those $b_i$ labeled DOWN forms an infinite strictly monotone decreasing sequence (all $\geq$’s in the definition of monotone decreasing are actually $>$’s).

So we have shown something slightly stronger than we set out to show: every infinite real sequence contains either an infinite monotone increasing subsequence or an infinite strictly monotone decreasing subsequence.
• Whichever operation has happened, we now have a subsequence \( b_1, a_{i_1}, a_{i_2}, a_{i_3}, \ldots, a_{i_{2n-3}} \), either with the property that \( b_1 \leq a_{i_k} \) for each \( k \) (if \( b_1 \) is labeled UP) or \( b_1 > a_{i_k} \) for each \( k \) (if \( b_1 \) is labeled DOWN).

• Set \( b_2 = a_{i_1} \), and repeat the process, exactly as in the infinite case. Because we are (at worst) halving the size of the sequence each time, we get to repeat at least \( 2n - 1 \) times. In the end, we get a subsequence \( b_1, b_2, b_3, \ldots, b_{2n-1} \), with each \( b_i \) labeled either UP or DOWN, and with the property that if \( b_i \) is labeled UP then \( b_i \leq b_j \) for all \( j > i \), and if \( b_i \) is labeled DOWN then \( b_i > b_j \) for all \( j > i \).

The sequence \( b_1, b_2, b_3, \ldots, b_{2n-1} \) is not necessarily monotone. However, by one more application of our pigeon-hole principle, at least one of the labels UP, DOWN must occur at least \( n \) times. If the label UP occurs \( n \) times, then the subsequence consisting of those \( b_i \) labeled UP forms a monotone increasing sequence of length \( n \); if the label DOWN occurs \( n \) times, then the subsequence consisting of those \( b_i \) labeled DOWN forms a strictly monotone decreasing sequence of length \( n \).

Again, we have shown something slightly stronger than we set out to show: every real sequence of length \( 2^{2n-1} \) contains either a monotone increasing subsequence of length \( n \) or a strictly monotone decreasing subsequence of length \( n \).

**A much better result**

What’s the smallest number \( f(n) \) such that every real sequence of length \( f(n) \) is certain to contain a subsequence of length \( n \) which is monotone? The argument of the last section shows that \( f(n) \leq 2^{2n-2} \). Erős and Szekeres proved the best possible result, showing that in truth \( f(n) \) is much smaller:

Every sequence of length \( (n - 1)^2 + 1 \) contains a subsequence of length \( n \) which is monotone.

See Theorem 2.4 of the textbook for a proof; homework problem 9 of Section 2.4 asks you to show that \( (n - 1)^2 + 1 \) cannot be replaced by \( (n - 1)^2 \).