P versus NP

Math 40210, Spring 2012

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Properties of graphs

A *property* of a graph is anything that can be described without referring to specific vertices

Examples of properties:

- being bipartite
- having an Euler circuit
- having a Hamilton cycle

Examples of non-properties:

- vertices 1, 2 and 3 form a triangle
- vertices $u$ and $v$ are in different components
The decision problem for a property

The decision problem for a particular property is to determine, for any possible input graph $G$, whether or not $G$ has the property

Examples of decision problems:

- Is $G$ bipartite?
- Does $G$ have an Euler circuit?
- Is $G$ Hamiltonian?
The class \( \mathbf{P} \)

A property is in class \( \mathbf{P} \) if there is a quick procedure for answering the decision problem, that works for every possible input graph (here “quick” formally means that the number of steps that the procedure takes is at most a polynomial in the number of vertices of the graph)

Examples of \( \mathbf{P} \) properties:

- Being bipartite: pick a vertex to put in \( X \); put its neighbors in \( Y \), put the neighbors of these new vertices in \( X \), etc.; wait until either the process finishes successfully (in which case \( G \) is bipartite), or it breaks down (in which case \( G \) has an odd cycle and is not bipartite)

- Having an Euler circuit: look at the vertex degrees; if they are all even, then \( G \) has an Euler circuit; if one or more is odd, then it does not

“\( \mathbf{P} \)” stands for “polynomial time”
The class **NP**

A property is *in class NP* if, for every graph $G$ for which the answer to the decision problem is YES, it is possible to present a quick proof of this fact (here again, “quick” formally means that the number of steps that need to be taken to verify that the proof is correct is at most a polynomial in the number of vertices of the graph)

Important note: the *proof* that $G$ has the property has to be short, but there’s no limit to the amount of time needed to come up with the proof

Another way to say this: a property is in class **NP** if whenever a graph $G$ has the property, I can quickly convince you that it has the property, *as long as I am given as much time as I need to prepare before beginning to convince you*

“**NP**” stands for “nondeterministic polynomial time”
Examples of **NP** properties

- Being bipartite: Before talking to you I find a valid bipartition $X \cup Y$, and I show it to you. **Or**: while you watch, I run the algorithm that answers the decision problem.

- Having an Euler circuit: Before talking to you I find an Euler circuit, and I show it to you. **Or**: while you watch, I run the algorithm that answers the decision problem.

- Any **P** property is also **NP**: to convince you that $G$ has the property, I run the (quick) algorithm that answers the decision problem.

- Having a Hamilton cycle: Before talking to you, I find a Hamilton cycle in the graph (this may take a **very** long time); once I have found it, I show it to you, and clearly you can quickly verify that it is indeed a Hamilton cycle.
The $1,000,000$ question

**P**: properties for which the decision problem can be quickly solved

**NP**: properties for which a YES answer to the decision problem can be quickly verified, given enough preparation time

We’ve seen that \( P \subseteq NP \). The $1,000,000$ question is this:

Is \( P = NP \)?

In other words, is it true that every decision problem for which a YES answer can be quickly verified, can also be quickly solved?

A specific example: is there a quick way of deciding whether a given graph has a Hamilton cycle?
The class **NP-complete**

A property is in the class **NP-complete** if a procedure for solving the decision problem for that property can be converted into a procedure for solving the decision problem for any other **NP** property, without any significant slow down.

**NP-complete** properties are in a sense the “hardest” properties: if you solve the decision problem for any one of them, you’ve solved the decision problem for all other **NP** properties.

Having a Hamilton cycle is known to be an **NP-complete** property. So:

- if you find a quick way to answer the question “does $G$ have a Hamilton cycle”, you’ve shown $P = NP$
- if you prove that no such quick way exists, you’ve shown $P \neq NP$
The Millennium Prize Problems

In 2000, the Clay Mathematics Institute identified seven important open problems in mathematics, and offered a prize of $1,000,000 for a solution to each one; see http://www.claymath.org/millennium/. One of these seven is the \( P \) versus \( NP \) problem.

The other six are problems of the form “prove \( X \)”, with the $1,000,000 only being offered for a proof of \( X \), not a counterexample. For \( P \) versus \( NP \), the full prize is guaranteed, whichever way the problem is resolved.