# Basic Combinatorics (Math 40210) Sec 01, Spring 2014, Quiz 1

Solutions

February 11, 2014

1. Give a proof of the *cancellation identity*: for all $n \geq k \geq 0$, $\binom{n+1}{k+1} = \frac{n+1}{k+1}\binom{n}{k}$.

   **Algebraic proof**:

$$\begin{aligned}\binom{n+1}{k+1} &= \frac{(n+1)!}{(k+1)!((n+1)-(k+1))!} \\ &= \frac{(n+1)n!}{(k+1)k!(n-k)!} \\ &= \frac{n+1}{k+1}\binom{n}{k}.\end{aligned}$$

   **Combinatorial proof**: We'll prove that $(k+1)\binom{n+1}{k+1} = (n+1)\binom{n}{k}$. The right-hand side counts the number of ways of selecting a committee of size $k+1$ from a group of $n+1$ people, with one member of the committee designated as the chair, by first selecting the chair from among the $n+1$ people ($n+1$ options), and then selecting the remaining $k$ members of the committee from among the remaining $n$ people ($\binom{n}{k}$ options). The left-hand side counts the same thing, by first selecting the $k+1$ members of the committee from among the $n+1$ people ($\binom{n+1}{k+1}$ options), and then selecting the chair from among the members of the committee ($k+1$ options).

2. Show that for every $n \geq 0$, and every integer $k$ (*positive or negative*),

$$\binom{n}{k}^2 \geq \binom{n}{k-1}\binom{n}{k+1}.$$

   **Solution**: A combinatorial proof of this is quite difficult, but an algebraic proof is easy. The left-hand side is a square, so always a least 0. If $k \leq 0$ then $\binom{n}{k-1} = 0$ and the right-hand side is 0; if $k \geq n$ then $\binom{n}{k+1} = 0$ and the left-hand side is 0; so the inequality is true for $k \leq 0$ and $k \geq n$. For $1 \leq k \leq n-1$,

$$\binom{n}{k}^2 \geq \binom{n}{k-1}\binom{n}{k+1}$$

   is the same as

$$\frac{n!n!}{k!(n-k)!k!(n-k)!} \geq \frac{n!n!}{(k-1)!(n-k+1)!(k+1)!(n-k-1)!}$$

   which is the same as

$$(k+1)(n-k+1) \geq k(n-k),$$

   which is true since $k+1 > k$ and $n-k+1 > n-k$. So the inequality is also true for $1 \leq k \leq n-1$.