15 Homework 6 (due April 1)  

Name: ____________

The purpose of this homework is to explore basics of graphs, including Eulerian trails and Hamiltonian paths/cycles.

Reading: Chapter 9 through Section 9.2.

1. • Part a): Prove that in any simple graph, there are two vertices that have the same degree as each other.

   Case i) No vertex of degree 0.
       Then if n vertices, possible degrees are 1, ..., n-1.
       By pigeon-hole principle, two vertices have same degree

   Case ii) Some vertex has degree 0; then one has degree n-1, so possible degrees are 0, ..., n-2, PHP works again

• Part b): Give an example of a multigraph in which there are not two vertices with the same degree.

   ![Graph example]
2. Part a): How many different simple (undirected, no multiple edges, no loops) graphs are there on vertex set $[n]$? (Remember that $[n] = \{1, \ldots, n\}$.) Your answer should not involve a summation.

\[ \binom{n}{2} \text{ possible edges; } 2 \binom{\binom{n}{2}}{2} \text{ choices for each possible edge: in or out?} \]

Part b): How many different directed graphs are there on vertex set $[n]$, if we do not allow multiple edges, but do allow loops (at most one loop per vertex)? **Clarification:** “no multiple edges” in this context means that we cannot have more than one copy of the edge from $a$ to $b$; but it is ok to have the edge from $a$ to $b$ (once) and the edge from $b$ to $a$ (once). Your answer should **not** involve a summation.

4 choices for each possible edge

\[ \binom{n}{2} \times 4 \]

2 choices for each vertex (loop or not)

\[ 2 \binom{\binom{n}{2}}{2} \]

Part c): A tournament $T$ on vertex set $[n]$ is a directed graph on $[n]$ such that for each pair of distinct elements $a, b \in [n]$, EITHER the edge from $a$ to $b$ is in $T$, OR the edge from $b$ to $a$, BUT NOT BOTH. (Think of a tournament as encoding the results of a round-robin tournament involving $n$ teams, in which ties are not allowed). How many different tournaments are there on vertex set $[n]$? Your answer should **not** involve a summation.

2 choices for each possible edge.

\[ \binom{n}{2} \times 2 \]
3. Part a): In a simple graph $G = (V, E)$, there is a walk that starts at vertex $x$ and ends at vertex $y$ ($x \neq y$). Prove that there is a path joining $x$ and $y$. [Refer to the handout on basic terminology to remind yourself of the definitions: basically, a walk allows repetition of vertices and edges, a path does not.]

Let $w = x_{0}v_{1}...v_{n}y_{n}$ be a shortest walk.

Claim: It is a path.

Proof: If a vertex name appears twice, can remove second occurrence and everything between the two occurrences to get a shorter walk, contradiction.

Part b): In a simple graph $G = (V, E)$, there is a closed walk of odd length. Prove that there is a cycle of odd length. [Again, refer to the handout on basic terminology; the length of a walk, path etc. is the number of edges in it.]

Consider shortest closed walk of odd length:

$\rightarrow$ If no repetitions of vertices, it's odd cycle, done.

$\rightarrow$ If vertex $v$ appears twice, walk breaks into two shorter closed walks one of which must be odd, contradiction; so this case can't occur, and we are done.

Walks $A$, $B$ can't both be even!
4. The \textit{n-dimensional hypercube} $Q_n$ is the graph whose vertex set is the set of all vectors $(x_1, \ldots, x_n)$ with each $x_i = 0$ or 1 (so there are $2^n$ vertices), with two vertices adjacent if the two vectors differ on exactly one coordinate. (So the graph $Q_2$ is just the usual square, and $Q_3$ is the familiar three dimensional cube). Here are pictures of the 0-, 1-, 2-, 3- and 4-dimensional hypercubes:

- Part a): What is the degree of each vertex in $Q_n$?

$$\n$$

- Part b): How many edges does $Q_n$ have?

$$\sum \text{ degrees } = 2(\# \text{ edges})$$

$$\therefore$$

$$2^n \cdot n$$

$$\leq_0 \# \text{ edges } = n 2^{n-1}$$
- Part c): For which $n$ does $Q_n$ have a closed Eulerian trail? Briefly justify your answer.

Need all degrees even and connected, so just need $n$ even

- Part d): Prove that for $n \geq 2$, $Q_n$ has a Hamiltonian cycle.

By induction on $n$, $n = 2$ easy

For $n > 2$:

Qn is composed of two copies of Qn-1, one "up", one "down", m edges between corresponding vertices (e.g., n = 3: 

By induction, there's HC in "up", so H path starting at a, ending at b, and there's a matching H path in "down" starting at a', ending at b'.

Get HC in Qn by $\rightarrow$ a to b along "up" H path

b to b'

b' to a' along "down" H path (backwards)

a to a'