Problem Solving in Math (Math 43900) Fall 2013

Week ten (November 5) problems — Games people play

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These problem are all about games played between two players. Usually when these problems appear in the Putnam competition, you are asked to determine which player wins when both players play as well as possible. Once you have decided which player wins (maybe based on analyzing small examples), you need to prove this in general. Often this entails demonstrating a *winning strategy*: for each possible move by the losing player, you can try to identify a single appropriate response for the winning player, such that if the winning player always uses these responses as the game goes on, then she will indeed win. It's important to remember that you must produce a response for the winning player for *every* possible move of the losing player — no just a select few.

The problems

1. A chocolate bar is made up of a rectangular m by n grid of small squares. Two players take turns breaking up the bar. On a given turn, a player picks a rectangular piece of chocolate and breaks it into two smaller rectangular pieces, by snapping along one whole line of subdivisions between its squares. The player who makes the last break wins. Does one of the players have a winning strategy for this game?

2. Two players, A and B, take turns naming positive integers, with A playing first. No player may name an integer that can be expressed as a linear combination, with positive integer coefficients, of previously named integers. The player who names "1" loses. Show that no matter how A and B play, the game will always end.

3. There are nine cards laid out on a table, numbered 1 through 9. Two players, A and B, take turns picking up cards (and once a card is picked up, it is out of play). As soon as one of the players has among his chosen cards three of them that sum to fifteen, that player wins.

   (a) If both players play perfectly, what happens?

   (b) What game are the players really playing?

4. Alan and Barbara play a game in which they take turns filling entries of an initially empty 1024 by 1024 array. Alan plays first. At each turn, a player chooses a real number and places it in a vacant entry. The game ends when all the entries are filled. Alan wins if the determinant of the resulting matrix is nonzero; Barbara wins if it is zero. Which player has a winning strategy?

5. Alice and Bob play a game in which they take turns removing stones from a heap that initially has n stones. The number of stones removed at each turn must be one less than a prime number. The winner is the player who takes the last stone. Alice plays first. Prove that there

are infinitely many n such that Bob has a winning strategy. (For example, if $n = 17$, then Alice might take 6 leaving 11; Bob might take 1 leaving 10; then Alice can take the remaining stones to win.)

6. A game starts with four heaps of beans, containing 3, 4, 5 and 6 beans. The two players move alternately. A move consists of taking either one bean from a heap, provided at least two beans are left behind in that heap, or a complete heap of two or three beans. The player who takes the last heap wins. Does the first or second player win? Give a winning strategy.