Problem Solving in Math (Math 43900) Fall 2013

Week ten solutions

Instructor: David Galvin

1. A chocolate bar is made up of a rectangular m by n grid of small squares. Two players take turns breaking up the bar. On a given turn, a player picks a rectangular piece of chocolate and breaks it into two smaller rectangular pieces, by snapping along one whole line of subdivisions between its squares. The player who makes the last break wins. Does one of the players have a winning strategy for this game?

   Solution: There is one piece of chocolate to start, and mn pieces at the end. Each turn by a player increases the number of squares by 1. Hence a game lasts $mn - 1$ turns, completely independently of the strategies of the two players! The winner is determined by the parity of m and n: if m and n are both odd, $mn - 1$ is even and player 2 wins. Otherwise $mn - 1$ is odd and player 1 wins.

   Source: I learned this from Peter Winkler, Dartmouth College.

2. Two players, A and B, take turns naming positive integers, with A playing first. No player may name an integer that can be expressed as a linear combination, with positive integer coefficients, of previously named integers. The player who names "1" loses. Show that no matter how A and B play, the game will always end.

   Solution: Suppose the first k moves consist of naming $x_1, \ldots, x_k$. Let g_k be the greatest common divisor of the x_i's. Consider the set of numbers expressible as a linear combination of the x_i's over positive integers. Each x in this set is an integer multiple of g_k (g_k divides the right-hand side of $x = \sum_i a_i x_i$, so it divides the left-hand side). We claim that there is some m such that all multiples of g_k greater than mg_k are in this set.

   If we can prove this claim, we are done. The sequence $(g_1, g_2, g_3, \ldots)$ is non-increasing. It stays constant in going from g_i to g_{i+1} exactly when x_{i+1} is a multiple of g_i, and drops exactly when x_{i+1} is not a multiple of g_i. By our claim, once the sequence has reached a certain g, it can only stay there for a finite length of time. So eventually that sequence becomes constantly 1. But once the sequence reaches 1, there are only finitely many numbers that can be legitimately played, and so eventually 1 must be played.

   Here's what we'll prove, which is equivalent to the claim: if $x_1, \ldots, x_k$ are relatively prime positive integers (greatest common divisor equals 1) then there exists an m such that all numbers greater than m can be expressed as a positive linear combination of the x_i's. We prove this by induction on k. When $k = 1$, $x_k = 1$ and the result is trivial. For $k > 1$, consider $x_1, \ldots, x_{k-1}$. These may not be relatively prime; say their greatest common divisor is d. By induction, there's an m' such that all positive integer multiples of d greater than $m'd$ can be expressed as a positive linear combination of the $x_1, \ldots, x_{k-1}$. Now d and x_k must be relatively prime (otherwise the x_i's would not be relatively prime), which means that there must be some positive integer e (which way may assume is between 1 and $x_k - 1$) with $ed \equiv 1$ (modulo x_k). If we add any multiple of x_k to e to get e', we still get $e'd \equiv 1$ (modulo x_k). Pick a multiple large enough that $e' > m'$. By induction, $e'd$ can be expressed as a positive integer combination of $x_1, \ldots, x_{k-1}$. So too can $2e'd, 3e'd, \ldots, x_k e'd$. These x_k numbers cover all the residue classes modulo x_k. Let m be one less than the largest of these numbers. For $\ell > m$, we can express ℓ as a positive linear combination of $x_1, \ldots, x_k$ as follows: first, determine the residue class of ℓ modulo x_k, say it's p. Then add the appropriate positive integer multiple of x_k to $pe'd$ (which can can be expressed as a positive integer combination of $x_1, \ldots, x_{k-1}$).

Source: This is the game of *Sylver coinage*, invented by John H. Conway; see `http://en.wikipedia.org/wiki/Sylver_coinage`. It is named after J. J. Sylvester, who proved that if a and b are relatively prime positive integers, then the largest positive integer that cannot be expressed as a positive linear combination of a and b is $(a-1)(b-1)-1$.

3. There are nine cards laid out on a table, numbered 1 through 9. Two players, A and B, take turns picking up cards (and once a card is picked up, it is out of play). As soon as one of the players has among his chosen cards three of them that sum to fifteen, that player wins.

   (a) If both players play perfectly, what happens?

   (b) What game are the players really playing?

   Solution: Here are the subsets of three distinct numbers that add to 15: $\{1,5,9\}$, $\{1,6,8\}$, $\{2,4,9\}$, $\{2,5,8\}$, $\{2,6,7\}$, $\{3,4,8\}$, $\{3,5,7\}$ and $\{4,5,6\}$. We can encode these eight triples as the rows, columns and diagonals of the following three-by-three array:

6	1	8
7	5	3
2	9	4

   So what the players are really doing is alternately selecting squares in a three-by-three array, with the object of choosing some set of three squares that fill up a row, column or diagonal. That is, they are playing tic-tac-toe, and the result of optimal play is a draw.

   Source: This escapes me at the moment.

4. Alan and Barbara play a game in which they take turns filling entries of an initially empty 1024 by 1024 array. Alan plays first. At each turn, a player chooses a real number and places it in a vacant entry. The game ends when all the entries are filled. Alan wins if the determinant of the resulting matrix is nonzero; Barbara wins if it is zero. Which player has a winning strategy?

   Solution: Barbara has a winning strategy. For example, Whenever Alan plays x in row i, Barbara can play $-x$ in some other place in row i (since there are an even number of places in row i, Alan will never place the last entry in a row if Barbara plays this strategy). So Barbara can ensure that all row-sums of the final matrix are 0, so that the column vector of all 1's is in the kernel of the final matrix, so it has determinant zero.

   Source: Putnam 2008 A2.

5. Alice and Bob play a game in which they take turns removing stones from a heap that initially has n stones. The number of stones removed at each turn must be one less than a prime number. The winner is the player who takes the last stone. Alice plays first. Prove that there are infinitely many n such that Bob has a winning strategy. (For example, if $n = 17$, then Alice might take 6 leaving 11; Bob might take 1 leaving 10; then Alice can take the remaining stones to win.)

   Solution: Suppose there are only finitely many n such that Bob will win if Alice starts with n stones, say all such $n < N$. Take $K > N$ so that $K - m + 1$ is composite for $m = 0, \ldots, N$. Starting with $n = K$, Alice must remove $p - 1$ stones, for p a prime number, leaving $m = K - p + 1$ stones. But $m > N$ since $K - m + 1 = p$ is prime and $K - m + 1$ is composite for $m < N$. By assumption, Alice can win starting from a heap of m stones. But it is Bob's turn to move, and so he could use the same strategy Alice would have used to win. This applies for any first move Alice could have made from a heap of K stones. Hence Bob has a winning strategy for a number $K > N$ of stones, contrary to hypothesis. Instead there must be infinitely many n for which Bob has a winning strategy.

   Source: Putnam 2006 A2; I've given the solution published in the American Mathematical Monthly verbatim. Implicit in this solution is the following useful fact: in a finite, two-person game with no draws allowed, one of the players must have a winning strategy.

6. A game starts with four heaps of beans, containing 3, 4, 5 and 6 beans. The two players move alternately. A move consists of taking either one bean from a heap, provided at least two beans are left behind in that heap, or a complete heap of two or three beans. The player who takes the last heap wins. Does the first or second player win? Give a winning strategy.

   Solution: The first player has a winning strategy. The first player wins by removing one bean from the pile of 3, leaving heaps of size 2, 4, 5, 6. Regarding heaps of size 2 and heaps of odd size as "odd", and heaps of even size other than 2 as "even", the total parity is now even. As long as neither player removes a single bean from a heap of size 3, the parity will change after each move. Thus the first player can always ensure that after his move, the total parity is even and there are no piles of size 3. (If the second player removes a heap of size 2, the first player can move in another odd heap; the resulting heap will be even and so cannot have size 3. If the second player moves in a heap of size greater than 3, the first player can move in the same heap, removing it entirely if it was reduced to size 3 by the second player.)

   Source: Putnam 1995 B5. I've reproduced the solution from *The William Lowell Putnam Mathematical Competition 1985–2000, Problems, Solutions, and Commentary* by Kiran S. Kedlaya, Bjorn Poonen and Ravi Vakil, where there is a further nice discussion.