Problem Solving in Math (Math 43900) Fall 2013

Week eleven (November 12) problems — Polynomials

Instructor: David Galvin

These problem are all about polynomials, which come up in virtually every Putnam competition.

Things to know about polynomials

- **Fundamental Theorem of Algebra**: Every polynomial $p(x) = x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_{n-1}x + a_n$, with real or complex coefficients, has a root in the complex numbers, that is, there is $c \in \mathbb{C}$ such that $p(c) = 0$.

- **Factorization**: In fact, every polynomial $p(x) = x^n + a_1x^{n-1} + a_2x^{n_2} + \ldots + a_{n-1}x + a_n$, with real or complex coefficients, has exactly n roots, in the sense that there is a vector $(c_1, \ldots, c_n)$ (perhaps with some repetitions) such that

  $$p(x) = (x - c_1)(x - c_2)\ldots(x - c_n).$$

  If a c appears in this vector exactly k times, it is called a *root* or *zero* of *multiplicity* k. The next bullet point gives a very useful consequence of this.

- **Two different polynomials of the same degree can't agree too often**: If $p(x)$ and $q(x)$ (over $\mathbb{R}$ or $\mathbb{C}$) both have degree at most n, and there are $n+1$ distinct numbers $x_1, \ldots, x_{n+1}$ such that $p(x_i) = q(x_i)$ for $i = 1, \ldots, n+1$, then $p(x)$ and $q(x)$ are equal for all x. [Because then $p(x) - q(x)$ is a polynomial of degree at most n with at least $n+1$ roots, so must be identically zero].

- **Complex conjugates**: If the coefficients of $p(x) = x^n + a_1x^{n-1} + a_2x^{n_2} + \ldots + a_{n-1}x + a_n$ are all **real**, then the complex roots occur in complex-conjugate pairs: if $s + it$ (with s, t real, and $i = \sqrt{-1}$) is a root, then $s - it$ is also a root.

- **Coefficients in terms of roots**: If $(c_1, \ldots, c_n)$ is the vector of roots of a polynomial $p(x) = x^n + a_1x^{n-1} + a_2x^{n_2} + \ldots + a_{n-1}x + a_n$ (over $\mathbb{R}$ or $\mathbb{C}$), then each of the coefficients can be expressed simply in terms of the roots: a_1 is the negative of the sum of the c_i's; a_2 is the sum of the products of the c_i's, taken two at a time, a_3 is the negative of the sum of the products of the c_i's, taken three at a time, etc. Concisely:

  $$a_k = (-1)^k \sum_{A \subseteq \{1,\ldots,n\},\ |A|=k} \prod_{i \in A} c_i.$$

- **Elementary symmetric polynomials**: The kth *elementary symmetric polynomial* in variables $x_1, \ldots, x_n$ is

  $$\sigma_k = \sum_{A \subseteq \{1,\ldots,n\},\ |A|=k} \prod_{i \in A} x_i$$

(these polynomials have already appeared in the last bullet point). A polynomial $p(x_1, \ldots, x_n)$ in n variables is *symmetric* if for every permutation π of $\{1, \ldots, n\}$, we have

$$p(x_1, \ldots, x_n) \equiv p(x_{\pi(1)}, \ldots, x_{\pi(n)}).$$

(For example, $x_1^2+x_2^2+x_3^2+x_4^2$ is symmetric, but $x_1^2+x_2^2+x_3^2+x_1x_4$ is not.) Every symmetric polynomial in variables $x_1, \ldots, x_n$ can be expressed as a linear combination of the σ_k's.

- **Some special values tell things about the coefficients**: (Rather obvious, but worth keeping in mind) If $p(x) = a_0x^n + a_1x^{n-1} + a_2x^{n_2} + \ldots + a_{n-1}x + a_n$, then

$$\begin{aligned} p(0) &= a_n \\ p(1) &= a_0 + a_1 + a_2 + \ldots + a_n \\ p(-1) &= a_n - a_{n-1} + a_{n-2} - a_{n-3} + \ldots + (-1)^n a_0. \end{aligned}$$

- **Intermediate value theorem**: If $p(x)$ is a polynomial with real coefficients (or in fact any continuous real function) such that for some $a < b$, $p(a)$ and $p(b)$ have different signs, then there is some c, $a < c < b$, with $p(c) = 0$.

- **Lagrange interpolation**: Suppose that $p(x)$ is a real polynomial of degree n, whose graph passes through the points (x_0, y_0), (x_1, y_1), …, (x_n, y_n). Then we can write

$$p(x) = \sum_{i=0}^{n} y_i \prod_{j \neq i} \frac{x - x_j}{x_i - x_j}.$$

- **Gauss' Lemma**: Let $p(x)$ be a monic polynomial of degree n with **integer** coefficients. (*Monic* means that the coefficient of x^n is 1.) If $p(a) = 0$ for rational a, then in fact a is an integer.

- **One more fact about integer polynomials**: Let $p(x)$ be a (not necessarily monic) polynomial of degree n with **integer** coefficients. For any integers a, b,

$$a - b | p(a) - p(b).$$

(So also,

$$p(a) - p(b) | p(p(a)) - p(p(b)),$$

etc.)

The problems

1. For which real values of p and q are the roots of the polynomial $x^3 - px^2 + 11x - q$ three successive (consecutive) integers? Give the roots in these cases.

2. (a) Determine all polynomials $p(x)$ such that $p(0) = 0$ and $p(x + 1) = p(x) + 1$ for all x.
   (b) Determine all polynomials $p(x)$ such that $p(0) = 0$ and $p(x^2 + 1) = (p(x))^2 + 1$ for all x.

3. Let$p(x) = a_nx_n+\ldots+a_1x+a_0$ be a polynomial with integer coefficients. If r is a rational root of $p(x)$ (i.e., if $p(r) = 0$), show that the numbers a_nr, $a_nr^2 + a_{n-1}r$, $a_nr^3 + a_{n-1}r^2 + a_{n-2}r$, …, $a_nr^n + a_{n-1}r^{n-1} + \ldots a_1r$ are all integers. **Note**: Don't assume Gauss' Lemma here.

4. Determine, with proof, all positive integers n for which there is a polynomial $p(x)$ of degree n satisfying the following three conditions:

   (a) $p(k) = k$ for $k = 1, 2, \ldots, n$,

   (b) $p(0)$ is an integer, and

   (c) $p(-1) = 2013$.

5. Find a non-zero polynomial $p(x, y)$ such that $p([t], [2t]) = 0$ for all real numbers t. (Here $[t]$ indicates the greatest integer less than or equal to t.)

6. Is there an infinite sequence $a_0, a_1, a_2, \ldots$ of nonzero real numbers such that for $n = 1, 2, 3, \ldots$ the polynomial

$$p_n(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$$

   has exactly n distinct real roots?