Problem Solving in Math (Math 43900) Fall 2013

Week eleven (November 12) solutions

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1. For which real values of p and q are the roots of the polynomial $x^3 - px^2 + 11x - q$ three successive (consecutive) integers? Give the roots in these cases.

   Solution: A polynomial with roots being three consecutive integers is of the form

   $$(x - (a - 1))(x - a)(x - (a + 1)) = x^3 - 3ax^2 + (3a^2 - 1)x - (a^3 - a)$$

   for some integer a. So, matching coefficients, we must have $3a^2 - 1 = 11$, or $a = \pm 2$. When $a = 2$ we get roots $1, 2, 3$ and $p = 6$, $q = 6$; when $a = -2$ we get roots $-3, -2, -1$ and $p = -6$, $q = -6$.

   Source: From a Harvey Mudd Putnam prep sheet.

2. (a) Determine all polynomials $p(x)$ such that $p(0) = 0$ and $p(x + 1) = p(x) + 1$ for all x.

   Solution: By induction, $p(x) = x$ for all positive integers x, so $p(x) - x$ is a polynomial with infinitely many zeros, so must be identically 0. We conclude that $p(x) = x$ is the only possible polynomial satisfying the given conditions.

   (b) Determine all polynomials $p(x)$ such that $p(0) = 0$ and $p(x^2 + 1) = (p(x))^2 + 1$ for all x.

   Solution: We have $p(0) = 0$, $p(1) = p(0)^2 + 1 = 1$, $p(2) = p(1)^2 + 1 = 2$, $p(5) = p(2)^2 + 1 = 5$, $p(26) = p(5)^2 + 1 = 26$ and in general, by induction, if the sequence (a_n) is defined recursively by $a_0 = 0$ and $a_{n+1} = a_n^2 + 1$, then $p(a_n) = a_n$. Since the sequence (a_n) is strictly increasing, we find that there are infinitely many distinct values x for which $p(x) = x$; as in the last part, this tells us that $p(x) = x$ is the only possible polynomial satisfying the given conditions.

   Source: Part b) was Putnam 1971 A2; part a) is a simpler instance of the same idea.

3. Let$p(x) = a_n x_n + \ldots + a_1 x + a_0$ be a polynomial with integer coefficients. If r is a rational root of $p(x)$ (i.e., if $p(r) = 0$), show that the numbers $a_n r$, $a_n r^2 + a_{n-1} r$, $a_n r^3 + a_{n-1} r^2 + a_{n-2} r$, $\ldots$, $a_n r^n + a_{n-1} r^{n-1} + \ldots a_1 r$ are all integers. **Note**: Don't assume Gauss' Lemma here.

   Solution/source: This was B1 of the 2004 Putnam; see the course website for a solution.

4. Determine, with proof, all positive integers n for which there is a polynomial $p(x)$ of degree n satisfying the following three conditions:

   (a) $p(k) = k$ for $k = 1, 2, \ldots, n$,

(b) $p(0)$ is an integer, and

(c) $p(-1) = 2013$.

Solution/source: This was modified from a UIUC mock Putnam; see `http://www.math.illinois.edu/~hildebr/putnam/problems/mock12sol.pdf` for a solution (replace "2012" everywhere in that solution with "2013", and "2013" with "2014").

5. Find a non-zero polynomial $p(x, y)$ such that $p([t], [2t]) = 0$ for all real numbers t. (Here $[t]$ indicates the greatest integer less than or equal to t.)

   Solution/source: One possibility is $p(x, y) = (2x - y)(2x - y + 1)$. Suppose $t = n + \alpha$, where n is an integer, and $0 \leq \alpha < 1$. If $\alpha < 1/2$, then $([t], [2t]) = (n, 2n)$, and $p([t], [2t]) = p(n, 2n) = 0$ (since $2x - y = 0$ in this case); if $\alpha \geq 1/2$, then $([t], [2t]) = (n, 2n + 1)$, and $p([t], [2t]) = p(n, 2n + 1) = 0$ (since $2x - y + 1 = 0$ in this case).

   Source: This was B1 of the 2005 Putnam.

6. Is there an infinite sequence $a_0, a_1, a_2, \ldots$ of nonzero real numbers such that for $n = 1, 2, 3, \ldots$ the polynomial

$$p_n(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$$

   has exactly n distinct real roots?

   Solution: We can explicitly construct such a sequence. Start with $a_0 = 1$ and $a_1 = -1$ (so case $n = 1$ works fine). We'll construct the a_i's inductively, always alternating in sign. Suppose we have $a_0, a_1, \ldots, a_{n-1}$. The polynomial $p_{n-1}(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_{n-1} x^{n-1}$ has real distinct roots $x_1 < \ldots < x_{n-1}$. Choose $y_1, \ldots, y_n$ so that

$$y_1 < x_1 < y_2 < x_2 < \ldots < y_{n-1} < x_{n-1} < y_n.$$

   The sequence $p_{n-1}(y_1), p_{n-1}(y_2), \ldots, p_{n-1}(y_n)$ alternates in sign (think about the graph of $y = p_{n-1}(x)$). As long as we choose a_n sufficiently close to 0, the sequence $p_n(y_1), p_n(y_2), \ldots, p_n(y_n)$ alternates in sign (this is by continuity). So, choose such an a_n. Now choose a y_{n+1} sufficiently large that $p_n(y_{n+1})$ has the opposite sign to $p_n(y_n)$ (this is where alternating the signs of the a_i's comes in — such a y_{n+1} exists exactly because a_n and a_{n-1} have opposite signs). We get that the sequence $p_n(y_1), p_n(y_2), \ldots, p_n(y_{n+1})$ alternates in sign. Hence $p_n(x)$ has n distinct real roots: one between y_1 and y_2, one between y_2 and y_3, etc., up to one between y_n and y_{n+1}. This accounts for all its root, and we are done.

   Source: Putnam 1990 B5.