Problem Solving in Math (Math 43900) Fall 2013

Week one (August 27) solutions

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1. A locker room has 100 lockers, numbered 1 to 100, all closed. I run through the locker room, and open every locker. Then I run through the room, and close the lockers numbered 2, 4, 6, etc. (all the even numbered lockers). Next I run through the room, and change the status of the lockers numbered 3, 6, 9, etc. (opening the closed ones, and closing the open ones). I keep going in this manner (on the ith run through the room, I change the status of lockers numbered i, $2i$, $3i$, etc.), until on my 100th run through the room I change the status of locker number 100 only. At the end of all this, which lockers are open?

   Solution: Locker n has its status changed once for each positive divisor of n, and so it is open in the end exactly if n has an odd number of positive divisors. If n has prime factorization $p_1^{a_1} \dots p_k^{a_k}$, then the number of positive divisors is $(a_1+1)\dots(a_k+1)$ (a positive divisor takes the form $p_1^{b_1} \dots p_k^{b_k}$ with $0 \leq b_i \leq a_i$ for each i; so there are a_i+1 choices for the value of b_i, with choices for different i's being independent). The product $(a_1+1)\dots(a_k+1)$ is odd only if each a_i+1 is odd, so only if each a_i is even. So Locker n has its status changed exactly when all the exponents in the prime factorization of n are even; in other words, exactly when n is a perfect square. So the open lockers are numbered 1, 4, 9, 16, 25, 36, 49, 64, 81 and 100.

   Source: This is an old, old problem. I learned it recently from my graduate student John Engbers.

2. Fix an integer $k \geq 1$. Let $f(x) = 1/(x^k-1)$. The nth derivative of $f(x)$ may be written as

$$f^{(n)}(x) = \frac{g_n(x)}{(x^k-1)^{n+1}}$$

   for some function $g_n(x)$. Find the value (in terms of n and k) of $g_n(1)$.

   Solution: The claimed form of the nth derivative of $f(x)$ is clearly correct: $g_n(x)$ can be found simply by differentiating $f(x)$ n times, and multipling the result by $(x^k-1)^{n+1}$. But notice that we can say a little more about $g_n(x)$, by doing the differentiations one at a time: for $n > 0$,

$$\begin{aligned}
f^{(n)}(x) &= \frac{d}{dx}\left(f^{(n-1)}(x)\right) \\
&= \frac{d}{dx}\left(\frac{g_{n-1}(x)}{(x^k-1)^n}\right) \\
&= \frac{d}{dx}\left(g_{n-1}(x)(x^k-1)^{-n}\right) \\
&= -nkx^{k-1}g_{n-1}(x)(x^k-1)^{-n-1} + g'_{n-1}(x)(x^k-1)^{-n} \\
&= \frac{-nkx^{k-1}g_{n-1}(x) + (x^k-1)g'_{n-1}(x)}{(x^k-1)^{n+1}}.
\end{aligned}$$

This constitutes a proof by induction of the following statement: the nth derivative of $f(x)$ may be written as

$$f^{(n)}(x) = \frac{g_n(x)}{(x^k-1)^{n+1}}$$

where the function $g_n(x)$ satisfies:

$$g_n(x) = \begin{cases} 1 & \text{if } n = 0 \\ -nkx^{k-1}g_{n-1}(x) + (x^k-1)g'_{n-1}(x) & \text{if } n > 0. \end{cases}$$

This makes $g_n(x)$ a polynomial in x (one can easily prove this by induction). That in particular means that $g'_{n-1}(x)$ is finite for all x and $n \geq 0$. So, evaluating the above recurrence at $x = 1$, we get:

$$g_n(1) = \begin{cases} 1 & \text{if } n = 0 \\ -nkg_{n-1}(1) & \text{if } n > 0 \end{cases}$$

(the term $(x^k-1)g'_{n-1}(x)$ disappearing at $x = 1$!). Now it is straightforward to prove by induction on n that

$$g_n(1) = (-1)^n k^n n!.$$

Source: This is (essentially) problem A1 from the 2002 Putnam Competition.

3. (a) Is there a sequence $(a_n)_{n\geq 1}$ of positive terms, such that both of the sums

$$\sum_{n=1}^{\infty} \frac{a_n}{n^3}, \qquad \sum_{n=1}^{\infty} \frac{1}{a_n}$$

converge?

Solution: Try $a_n = n^\alpha$. Then

$$\sum_{n=1}^{\infty} \frac{a_n}{n^3} = \sum_{n=1}^{\infty} \frac{1}{n^{3-\alpha}}$$

which converges as long as $3 - \alpha > 1$ or $\alpha < 2$; and

$$\sum_{n=1}^{\infty} \frac{1}{a_n} = \sum_{n=1}^{\infty} \frac{1}{n^\alpha}$$

which converges as long as $\alpha > 1$. So $a_n = n^\alpha$ for any α satisfying $1 < \alpha < 2$ will work.

(b) Is there a sequence $(a_n)_{n\geq 1}$ of positive terms, such that both of the sums

$$\sum_{n=1}^{\infty} \frac{a_n}{n^2}, \qquad \sum_{n=1}^{\infty} \frac{1}{a_n}$$

converge?

Solution: This time $a_n = n^\alpha$ won't work; we need $\alpha < 1$ for the first sum to converge, and $\alpha > 1$ for the second. This suggests that no such a_n exists, and here's a proof of that fact:

Assume that both sums converge. Then so does

$$\sum_{n=1}^{\infty}\left(\frac{a_n}{n^2}+\frac{1}{a_n}\right).$$

But this seems fishy: if a_n is small, then $1/a_n$ is large; and if a_n is large, then a_n/n^2 is large. So it seems that the summand is always large, making convergence hard. We can easily formalize this: if $a_n \geq n$ then $a_n/n^2 \geq 1/n$, and if $a_n \leq n$ then $1/a_n \geq 1/n$. So

$$\sum_{n=1}^{\infty}\left(\frac{a_n}{n^2}+\frac{1}{a_n}\right) > \sum_{n=1}^{\infty}\frac{1}{n} = \infty,$$

a contradiction.

Note: If we allow negative a_n, then it is possible to get both sums to converge: take, for example, $a_n = (-1)^n n$.

Source: I found this problem in the lovely book *A Mathematical Orchard, Problems and Solutions* by Krusemeyer, Gilbert and Larson.

4. Prove the following statement: for every even number $n \geq 2$, the numbers 1 up to n can be paired off (into $n/2$ pairs) in such a way that the sum of each pair is a prime number (for example, if $n = 8$ the pairing scheme $\{1,2\}$, $\{3,4\}$, $\{5,6\}$, $\{7,8\}$ doesn't work, because $7+8=15$ is not prime; but the pairing scheme $\{1,4\}$, $\{2,3\}$, $\{5,8\}$, $\{6,7\}$ does).

   Solution: Proof by (strong) induction on n. For $n = 2$ the statement is obvious. Consider $n > 2$. By Bertrand's Postulate, there is a prime number p with $n < p < 2n$. Pair the numbers $(n, p-n)$, $(n-1, p-n+1)$, $(n-2, p-n+2)$, etc., down to $((p+1)/2, (p-1)/2)$; all these (distinct and disjoint) pairs sum to p. (E.g., if $n = 10$, and $p = 13$, we would pair $(10,3)$, $(9,4)$, $(8,5)$ and $(7,6)$). This leaves the numbers from 1 to $p-n-1$ to be paired; since $p-n-1$ is even and less than n, this pairing can be done by induction!

   Source: This beautiful argument appeared in:

   Greenfield, L. and Greenfield, S., Some problems of combinatorial number theory related to Bertrands postulate, *J. Integer Seq.* **1** (1998), Article 98.1.2.

5. Find the sum of the digits, of the sum of the digits, of the sum of the digits, of the number 4444^{4444}.

   Solution: We start with

   $$4444^{4444} < 10000^{10000} = 10^{40000}.$$

   Among all numbers below 10^{40000}, none has a larger sum of digits than $10^{40000}-1$ (a string of 40000 9's). So the sum of the digits of 4444^{4444} is at most $9 \times 40000 < 1000000$. Among all numbers below 1000000, none has a larger sum of digits than 999999. So the sum of the digits of the sum of the digits of 4444^{4444} is at most 54. Among all numbers at most 54, none has a larger sum of digits than 49. So the sum of the digits of the sum of the digits of the sum of the digits of 4444^{4444} is at most 13.

   Now we use a useful fact: the remainder of a number, on division by 9, is the same as the remainder of the sum of the digits on division by 9. To prove this, we digress into modular

arithmetic. Write $a \equiv b \pmod{k}$ if a and b leave the same remainder on division by k (so, e.g., $13 \equiv 4 \pmod{3}$ because they both leave remainder 1, but $13 \not\equiv 4 \pmod{5}$, since 13 leaves a remainder of 3 and 4 a remainder of 4.) Here are two easy facts:

Fact 1: if $a \equiv b \pmod{k}$ and $c \equiv d \pmod{k}$ then $a + c \equiv b + d \pmod{k}$

and

Fact 2: if $a \equiv b \pmod{k}$ and $c \equiv d \pmod{k}$ then $ac \equiv bd \pmod{k}$.

Now look at a number $a = a_\ell a_{\ell-1} \dots a_2 a_1$ in decimal. We have

$$a = a_\ell 10^{\ell-1} + a_{\ell-1} 10^{\ell-2} + \dots a_2 10 + a_1.$$

Since $10 \equiv 1 \pmod{9}$ we have (by Fact 2) $10^i \equiv 1 \pmod{9}$ for all i, and so (again by Fact 2) $a_i 10^{i-1} \equiv a_i \pmod{9}$. By Fact 1, we then have

$$a_\ell 10^{\ell-1} + a_{\ell-1} 10^{\ell-2} + \dots a_2 10 + a_1 \equiv a_\ell + a_{\ell-1} + \dots + a_2 + a_1 \pmod{9}.$$

In other words, the remainder of a number, on division by 9, is the same as the remainder of the sum of the digits on division by 9, as claimed.

This fact implies that the sum of the digits of the sum of the digits of the sum of the digits of 4444^{4444} leaves the same remainder on division by 9 as 4444^{4444} itself does.

To calculate the remainder of 4444^{4444} on division by 9, we can use a repeated multiplication trick. It's easy that

$$4444 \equiv 7 \pmod{9}.$$

By Fact 2:

$$\begin{aligned}
4444^2 &\equiv 49 \equiv 4 \pmod{9} \\
4444^4 &\equiv 16 \equiv 7 \pmod{9} \\
4444^8 &\equiv 49 \equiv 4 \pmod{9} \\
4444^{16} &\equiv 16 \equiv 7 \pmod{9} \\
4444^{32} &\equiv 49 \equiv 4 \pmod{9} \\
4444^{64} &\equiv 16 \equiv 7 \pmod{9} \\
4444^{128} &\equiv 49 \equiv 4 \pmod{9} \\
4444^{256} &\equiv 16 \equiv 7 \pmod{9} \\
4444^{512} &\equiv 49 \equiv 4 \pmod{9} \\
4444^{1024} &\equiv 16 \equiv 7 \pmod{9} \\
4444^{2048} &\equiv 49 \equiv 4 \pmod{9} \\
4444^{4096} &\equiv 16 \equiv 7 \pmod{9}.
\end{aligned}$$

By Fact 2 again:

$$4444^{4444} = 4444^{4096} 4444^{256} 4444^{64} 4444^{16} 4444^{8} 4444^{4} \equiv 7.7.7.7.4.7 \equiv 7 \pmod{9}.$$

So 4444^{4444} leaves a remainder of 7 on division by 9, and also the sum of the digits of the sum of the digits of the sum of the digits of 4444^{4444} leaves a remainder of 7 on division by 9; but we've calculated that this last is at most 13. The only number at most 13 that leaves

a remainder of 7 on division by 9 is 7 it self; so the sum of the digits of the sum of the digits of the sum of the digits of 4444^{4444} must be 7.

Source: This was IMO (International Mathematical Olympiad) 1975 problem 4. Funnily enough, if you had got this question fully correct at the IMO, you would have scored 7 points!

6. Find polynomials $f(x)$, $g(x)$ and $h(x)$ such that

$$|f(x)| - |g(x)| + h(x) = \begin{cases} -1 & \text{if } x < -1, \\ 3x + 2 & \text{if } -1 \le x \le 0, \\ -2x + 2 & \text{if } x > 0, \end{cases}$$

OR show that no such polynomials can be found.

Solution: This was on the 1999 Putnam (problem A1); see the appropriate Putnam book on reserve in the library for a solution.

Basically: Since $|f(x)|-|g(x)|+h(x)$ is piecewise linear, it seems reasonable to assume that if such polynomials exist, they are all linear. Since f and g appear inside absolute value signs, we might as well assume that they both have positive leading coefficients. There's a kink at $x = -1$, so one of f, g must change sign at -1, and a kink at 0, so the other changes sign at 0. If f changed sign at 0 and g at -1, then we would expect $|f(x)| - |g(x)| + h(x)$ to start *decreasing* as it crossed -1, and *increasing* as it crossed 0; but it does the opposite. To get this behavior, we should have f changing sign at -1 and g at 0. So, if an f, g, h of the king described exist, they satisfy three linear equations, obtained by thinking about the three regimes $(-\infty, -1)$, $(-1, 0)$ and $(0, \infty)$:

$$\begin{aligned} -f(x) + g(x) + h(x) &= -1 \\ f(x) + g(x) + h(x) &= 3x + 2 \\ f(x) - g(x) + h(x) &= -2x + 2. \end{aligned}$$

Solving yields a valid solution:

$$f(x) = \frac{3x+3}{2}, \qquad g(x) = \frac{5x}{2}, \qquad h(x) = -x + \frac{1}{2}.$$

7. Consider the following game played with a deck of $2n$ cards, numbered from 1 to $2n$. The deck is randomly shuffled and n cards are dealt to each of two players, A and B. Beginning with A, the players take turns discarding one of their remaining cards and announcing its number. The game ends as soon as the sum of the numbers on the discarded cards is divisible by $2n + 1$. The last person to discard wins the game.

   Assuming optimal strategy by both A and B, who wins?

Solution: This was on the 1993 Putnam (problem B2); see the appropriate Putnam book on reserve in the library for a solution.

Basically: B wins. Think about A throwing down his first card, x. We may assume that A also has $2n+1-x$, otherwise B has it and wins immediately. Consider two cards that B has, say y and z. It must be that $x+y$ and $x+z$ leave a different remainder on division by $2n+1$ (if they left the same remainder, so would y and z on their own, impossible). So each of y

and z would require a different response from A, if A were going to win on his next throw. In other words, B can associate to each of n cards a different number, and say to himself "if I play card y, A must play card number(y) to win on his next throw". But A only has $n-1$ cards, so there must be a card y in B's hand such that A does not have number(y). B plays that card, thus ensuring that the game continues at least until B gets to make his second throw. But then the same type of argument applies, and B can keep putting off the end of the game, until finally B has just one card left, which he throws down to win.

8. Let f be a non-constant polynomial with positive integer coefficients. Prove that if n is a positive integer, then $f(n)$ divides $f(f(n)+1)$ if and only if $n=1$.

   Solution: This was on the 2007 Putnam (problem B1); see the appropriate link on the course website for a solution.

   Basically: We'll use modular arithmetic. Write $f(x) = \sum_{k=0}^{m} c_k x^k$. We have

$$f(f(n)+1) = \sum_{k=0}^{m} c_k (f(n)+1)^k \equiv \sum_{k=0}^{m} c_k = f(1) \pmod{f(n)}. \tag{1}$$

   Here we're using Fact 2 from earlier ($f(n)+1 \equiv 1 \pmod{f(n)}$) implies $c_k(f(n)+1)^k \equiv c_k(1)^k = c_k \pmod{f(n)}$), and then Fact 1 to add these all together.

   Plugging $n=1$ into (1) gives

$$f(f(1)+1) \equiv f(1) \equiv 0 \pmod{f(1)},$$

   i.e., $f(1) | f(f(1)+1)$.

   For $n>1$, could we have

$$f(f(n)+1) \equiv 0 \pmod{f(n)}$$

   (i.e., $f(n) | f(f(n)+1)$)? Combining with (1), this would say

$$f(1) \equiv 0 \pmod{f(n)},$$

   i.e., $f(n) | f(1)$. But for $n>1$, we have $f(n) > f(1) > 0$ (we're working with a non-constant polynomial with positive coefficients), so we can't have $f(n) | f(1)$.

   Note: When this problem appeared on the Putnam Competition, the requirement that f be non-constant was not mentioned. Without it, the statement to be proved is false. This is a cautionary note: sometimes (*extremely* rarely), because of an oversight like this, you have to make a small tweak to the problem statement to get at the real problem.