Problem Solving in Math (Math 43900) Fall 2013

Week two (September 3) problems — induction

Instructor: David Galvin

Induction

Suppose that $P(n)$ is an assertion about the natural number n. *Induction* is essentially the following: if there is some a for which $P(a)$ is true, and if for all $n \geq a$ we have that the truth of $P(n)$ implies the truth of $P(n+1)$, then we can conclude that $P(n)$ is true for all $n \geq a$.

Induction works because of the fundamental fact that a non-empty subset of the natural numbers must have a least element. To see why this lets induction work, suppose that we know that $P(a)$ is true for some a, and that we can argue that for all $n \geq a$, the truth of $P(n)$ implies the truth of $P(n+1)$. Suppose further that there are some $n \geq a$ for which $P(n)$ is **not** true. Let $F = \{n | n \geq a,\ P(n) \text{ not true}\}$. By assumption F is non-empty, so has a least element, n_0 say. We know $n_0 \neq a$, since $P(a)$ **is** true; so $n_0 \geq a+1$. That means that $n_0 - 1 \geq a$, and since $n_0 - 1 \notin F$ (if it was, n_0 would not be the least element) we know $P(n_0 - 1)$ **is** true. But then, by assumption, $P((n_0 - 1) + 1) = P(n_0)$ **is** true, a contradiction!

Example: Prove that a set of size $n \geq 1$ has 2^n subsets (including the empty set and the set itself).

Solution: Let $P(n)$ be the statement "a set of size n has 2^n subsets". We prove that $P(n)$ is true for all $n \geq 1$ by induction. We first establish a *base case*. When $n = 1$, the generic set under consideration is $\{x\}$, which has $2 = 2^1$ subsets ($\{x\}$ and $\emptyset$); so $P(1)$ is true.

Next we establish the *inductive step*. Suppose that for some $n \geq 1$, $P(n)$ is true. Consider $P(n+1)$. The generic set under consideration now is $\{x_1, \ldots, x_n, x_{n+1}\}$. We can construct a subset of $\{x_1, \ldots, x_n, x_{n+1}\}$ by first forming a subset of $\{x_1, \ldots, x_n\}$, and then either adding the element x_{n+1} to this subset, or not. This tells us that the number of subsets of $\{x_1, \ldots, x_n, x_{n+1}\}$ is 2 times the number of subsets of $\{x_1, \ldots, x_n, \}$. Since $P(n)$ is assumed true, we know that $\{x_1, \ldots, x_n, \}$ has 2^n subsets (this step is usually referred to as applying the *inductive hypothesis*); so $\{x_1, \ldots, x_n, x_{n+1}\}$ has $2 \times 2^n = 2^{n+1}$ subsets. This shows that the truth of $P(n)$ implies that of $P(n+1)$, and the proof by induction is complete.

Strong Induction

Induction is a great tool because it gives you somewhere to start from in an argument. And sometimes, the more you start with, the further you'll go. That's why the principle of *Strong Induction* is worth keeping in mind: if there is some a for which $P(a)$ is true, and if for each $n > a$ we have that the truth of $P(m)$ for all m, $a \leq m < n$, implies the truth of $P(n)$, then we can conclude that $P(n)$ is true for all $n \geq a$.

The proof that this works is almost the same as the proof that induction works. What's good about strong induction is that when you are at the part of the argument where you have to show that the truth of $P(n+1)$ from some assumptions about earlier assertions, you now have a lot more to work with: each of $P(a), P(a+1), \ldots, P(n-1)$, rather than just $P(n)$ alone. Sometimes this is helpful, and sometimes it's absolutely necessary.

Example: Prove that every integer $n \geq 2$ can be written as $n = p_1 \ldots p_\ell$ where the p_i's are (not necessarily distinct) prime numbers.

Solution: Let $P(n)$ be the statement: "n can be written as $n = p_1 \ldots p_\ell$ where the p_i's are (not necessarily distinct) prime numbers". We'll prove that $P(n)$ is true for all $n \geq 2$ by strong induction.

$P(2)$ is true, since $2 = 2$ works.

Now consider $P(n)$ for some $n > 2$. We want to show how the (simultaneous) truth of $P(2), \ldots, P(n-1)$ implies the truth of $P(n)$. If n is prime, then $n = n$ works to show that $P(n)$ holds. If n is not a prime, then its composite, so $n = ab$ for some numbers a, b with $2 \leq a < n$ and $2 \leq b < n$. We're allowed to assume that $P(a)$ and $P(b)$ are true, that is, that $a = p_1 \ldots p_\ell$ where the p_i's are (not necessarily distinct) prime numbers, and that $a = q_1 \ldots q_m$ where the q_i's are (not necessarily distinct) prime numbers. It follows that

$$n = ab = p_1 \ldots p_\ell q_1 \ldots q_m.$$

This is a product is (not necessarily distinct) prime numbers, and so $P(n)$ is true.

So, by strong induction, we conclude that $P(n)$ is true for all $n \geq 2$.

Notice that we would have gotten exactly nowhere with this argument if, in trying to prove $P(n)$, all we had been allowed to assume was $P(n-1)$.

Recurrences

Sometimes we are either given a sequence of numbers via a recurrence relation, or we can argue that there is such relation that governs the growth of a sequence. A sequence $(b_n)_{n \geq a}$ is defined via a *recurrence relation* if some initial values, $b_a, b_{a+1}, \ldots, b_k$ say, are given, and then a rule is given that allows, for each $n > k$, b_n to be computed as long as we know the values $b_a, b_{a+1}, \ldots, b_{n-1}$.

Sequences defined by a recurrence relation, and proofs by induction, go hand-in-glove. We may have more to say about recurrence relations later in the semester, but for now, we'll confine ourselves to an illustrative example.

Example: Let a_n be the number of different ways of covering a 1 by n strip with 1 by 1 and 1 by 3 tiles. Prove that $a_n < (1.5)^n$.

Solution: We start by figuring out how to calculate a_n via a recurrence. Some initial values of a_n are easy to compute: for example, $a_1 = 1$, $a_2 = 1$ and $a_3 = 2$. For $n \geq 4$, we can tile the 1 by n strip EITHER by first tiling the initial 1 by 1 strip with a 1 by 1 tile, and then finishing by tiling the remaining 1 by $n-1$ strip in any of the a_{n-1} admissible ways; OR by first tiling the initial 1 by 3 strip with a 1 by 3 tile, and then finishing by tiling the remaining 1 by $n-3$ strip in any of the a_{n-3} admissible ways. It follows that for $n \geq 4$ we have $a_n = a_{n-1} + a_{n-3}$. So a_n (for $n \geq 1$) is determined by the recurrence

$$a_n = \begin{cases} 1 & \text{if } n = 1, \\ 1 & \text{if } n = 2, \\ 2 & \text{if } n = 3, \text{ and} \\ a_{n-1} + a_{n-3} & \text{if } n \geq 4. \end{cases}$$

Notice that this gives us enough information to calculate a_n for all $n \geq 1$: for example, $a_4 = a_3 + a_1 = 3$, $a_5 = a_4 + a_2 = 4$, and $a_6 = a_5 + a_3 = 6$.

Now we prove, by strong induction, that $a_n < 1.5^n$. That $a_1 = 1 < 1.5^1$, $a_2 = 1 < (1.5)^2$ and $a_3 = 2 < (1.5)^3$ is obvious. For $n \geq 4$, we have

$$\begin{aligned} a_n &= a_{n-1} + a_{n-3} \\ &< (1.5)^{n-1} + (1.5)^{n-3} \\ &= (1.5)^n \left(\frac{2}{3} + \left(\frac{2}{3}\right)^3 \right) \\ &= (1.5)^n \left(\frac{26}{27} \right) \\ &< (1.5)^n, \end{aligned}$$

(the second line using the inductive hypothesis) and we are done by induction.

Notice that we really needed strong induction here, and we really needed all three of the base cases $n = 1, 2, 3$ (think about what would happen if we tried to use regular induction, or what would happen if we only verified $n = 1$ as a base case); notice also that an induction argument can be written quite concisely, while still being fully correct, without fussing too much about "$P(n)$".

The problems

1. Let $f(n)$ be the number of regions which are formed by n lines in the plane, where no two lines are parallel and no three meet in a point (e.g. $f(1) = 2$, $f(2) = 4$ and $f(3) = 7$). Find a formula for $f(n)$, and prove that it is correct.

2. Prove the following inequalities:

   (a) $2(\sqrt{n+1} - 1) \leq 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots + \frac{1}{\sqrt{n}} \leq 2\sqrt{n}$.

   (b) $\prod_{k=1}^{n} (2k)! \geq ((n+1)!)^n$.

3. Define a sequence $(a_n)_{n \geq 1}$ by

   $$a_1 = 1, \qquad a_{2n} = a_n, \qquad \text{and} \qquad a_{2n+1} = a_n + 1.$$

   Prove that a_n counts the number of 1's in the binary representation of n.

4. Prove that for all $n \geq 2$, it is possible to write $n! - 1$ as the sum of $n - 1$ numbers, each of which is a divisor of $n!$.

5. Find (with proof!) all sequences $(a_n)_{n \geq 0}$ of positive real numbers for which

   $$\frac{a_1 + 2a_2 + 3a_3 + \ldots + ka_k}{a_1 + a_2 + \ldots + a_k} = \frac{k+1}{2}$$

   for all $k \geq 1$.

6. Define a sequence $(a_n)_{n \geq 0}$ by

   $$a_n = \begin{cases} 9 & \text{if } n = 0, \text{ and} \\ 3a_{n-1}^4 + 4a_{n-1}^3 & \text{if } n \geq 1. \end{cases}$$

   Prove that for all $n \geq 0$, a_n ends with at least 2^n 9's in its decimal representation.

7. Prove if $a_1, \ldots, a_n$ are positive reals, then

$$\frac{a_1 + \ldots + a_n}{n} \geq (a_1 \ldots a_n)^{1/n}.$$

8. You are given a 64 by 64 chessboard, and 1365 L-shaped tiles (2 by 2 tiles with one square removed). One of the squares of the chessboard is painted purple. Is is possible to tile the chessboard using the given tiles, leaving only the purple square exposed?